Chapter III. Elementary Properties and Examples of Analytic Functions

III.3. Analytic Functions as Mappings, Möbius Transformations—Proofs
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Theorem III.3.4

Theorem III.3.4. If \( f : G \rightarrow \mathbb{C} \) is analytic then \( f \) preserves angles at each point \( z_0 \) of \( G \) where \( f'(z_0) \neq 0 \).

Proof. Suppose \( \gamma \) is a smooth path in a region \( G \) and \( f : G \rightarrow \mathbb{C} \) is analytic. Then \( \sigma = f \circ \gamma \) is smooth and \( \sigma'(t) = f'(%(\gamma(t)))\gamma'(t) \).
Theorem III.3.4

Theorem III.3.4. If $f : G \to \mathbb{C}$ is analytic then $f$ preserves angles at each point $z_0$ of $G$ where $f'(z_0) \neq 0$.

Proof. Suppose $\gamma$ is a smooth path in a region $G$ and $f : G \to \mathbb{C}$ is analytic. Then $\sigma = f \circ \gamma$ is smooth and $\sigma'(t) = f'(\gamma(t))\gamma'(t)$. Let $z_0 = \gamma(t_0)$ and suppose $\gamma'(t_0) \neq 0$ and $f'(z_0) \neq 0$. 
Theorem III.3.4. If \( f : G \to \mathbb{C} \) is analytic then \( f \) preserves angles at each point \( z_0 \) of \( G \) where \( f'(z_0) \neq 0 \).

**Proof.** Suppose \( \gamma \) is a smooth path in a region \( G \) and \( f : G \to \mathbb{C} \) is analytic. Then \( \sigma = f \circ \gamma \) is smooth and \( \sigma'(t) = f'(\gamma(t))\gamma'(t) \). Let \( z_0 = \gamma(t_0) \) and suppose \( \gamma'(t_0) \neq 0 \) and \( f'(z_0) \neq 0 \).
Theorem III.3.4. If \( f : G \rightarrow \mathbb{C} \) is analytic then \( f \) preserves angles at each point \( z_0 \) of \( G \) where \( f'(z_0) \neq 0 \).

Proof. Suppose \( \gamma \) is a smooth path in a region \( G \) and \( f : G \rightarrow \mathbb{C} \) is analytic. Then \( \sigma = f \circ \gamma \) is smooth and \( \sigma'(t) = f'(\gamma(t))\gamma'(t) \). Let \( z_0 = \gamma(t_0) \) and suppose \( \gamma'(t_0) \neq 0 \) and \( f'(z_0) \neq 0 \).

\[
\sigma'(t_0) = f'(\gamma(t_0))\gamma'(t_0)
\]

Then \( \sigma'(t_0) = f'(\gamma(t_0))\gamma'(t_0) \neq 0 \) and
\[
\arg(\sigma'(t_0)) = \arg(f'(\gamma(t_0))) + \arg(\gamma'(t_0)). \quad (*)
\]
Theorem III.3.4. If \( f : G \rightarrow \mathbb{C} \) is analytic then \( f \) preserves angles at each point \( z_0 \) of \( G \) where \( f'(z_0) \neq 0 \).

**Proof.** Suppose \( \gamma \) is a smooth path in a region \( G \) and \( f : G \rightarrow \mathbb{C} \) is analytic. Then \( \sigma = f \circ \gamma \) is smooth and \( \sigma'(t) = f'(\gamma(t))\gamma'(t) \). Let \( z_0 = \gamma(t_0) \) and suppose \( \gamma'(t_0) \neq 0 \) and \( f'(z_0) \neq 0 \).

Then \( \sigma'(t_0) = f'(\gamma(t_0))\gamma'(t_0) \neq 0 \) and

\[
\arg(\sigma'(t_0)) = \arg(f'(\gamma(t_0))) + \arg(\gamma'(t_0)).
\] (*
Theorem III.3.4. If $f : G \rightarrow \mathbb{C}$ is analytic then $f$ preserves angles at each point $z_0$ of $G$ where $f'(z_0) \neq 0$.

Proof (continued). So if $\gamma_1$ and $\gamma_2$ are smooth paths which intersect at $x_0$ and $\gamma'_1(t_1) \neq 0 \neq \gamma'_2(t_2)$, then $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$ are smooth. So (*) implies

$$\arg(\sigma'_1(t_1)) - \arg(\sigma'_2(t_2)) = \arg(f'(\gamma_1(t_1)) + \arg(\gamma'_1(t_1))$$

$$-\{\arg(f'(\gamma_2(t_2)) + \arg(\gamma'_2(t_2)))\} = \arg(\gamma'_1(t_1)) - \arg(\gamma'_2(t_2)).$$
Theorem III.3.4. If \( f : G \rightarrow \mathbb{C} \) is analytic then \( f \) preserves angles at each point \( z_0 \) of \( G \) where \( f'(z_0) \neq 0 \).

Proof (continued). So if \( \gamma_1 \) and \( \gamma_2 \) are smooth paths which intersect at \( x_0 \) and \( \gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2) \), then \( \sigma_1 = f \circ \gamma_1 \) and \( \sigma_2 = f \circ \gamma_2 \) are smooth. So (*) implies

\[
\arg(\sigma_1'(t_1)) - \arg(\sigma_2'(t_2)) = \arg(f'(\gamma_1(t_1)) + \arg(\gamma_1'(t_1)) \\
- \{\arg(f'(\gamma_2(t_2)) + \arg(\gamma_2'(t_2))\} = \arg(\gamma_1'(t_1)) - \arg(\gamma_2'(t_2)).
\]

That is, an angle between \( \gamma_1 \) and \( \gamma_2 \) at \( z_0 \) is the same as the angle between \( \sigma_1 = f \circ \gamma_1 \) and \( \sigma_2 = f \circ \gamma_2 \). \( \square \)
Theorem III.3.4. If $f : G \to \mathbb{C}$ is analytic then $f$ preserves angles at each point $z_0$ of $G$ where $f'(z_0) \neq 0$.

Proof (continued). So if $\gamma_1$ and $\gamma_2$ are smooth paths which intersect at $x_0$ and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$, then $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$ are smooth. So (*) implies

$$\arg(\sigma_1'(t_1)) - \arg(\sigma_2'(t_2)) = \arg(f'(\gamma_1(t_1)) + \arg(\gamma_1'(t_1))$$

$$-\{\arg(f'(\gamma_2(t_2)) + \arg(\gamma_2'(t_2))\} = \arg(\gamma_1'(t_1)) - \arg(\gamma_2'(t_2)).$$

That is, an angle between $\gamma_1$ and $\gamma_2$ at $z_0$ is the same as the angle between $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$. \qed
Proposition III.3.8. If \( z_2, z_3, z_4 \in \mathbb{C}_\infty \) are distinct, and \( T \) is a Möbius transformation then \((z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)\) for any \( z_1 \in \mathbb{C}_\infty \).

**Proof.** Let \( S(z) = (z, z_1, z_2, z_3, z_4) \) (as defined above). Then \( S \) is a Möbius transformation. Define \( M = S \circ T^{-1} \).
**Proposition III.3.8.** If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct, and $T$ is a Möbius transformation then $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any $z_1 \in \mathbb{C}_\infty$.

**Proof.** Let $S(z) = (z, z_1, z_2, z_3, z_4)$ (as defined above). Then $S$ is a Möbius transformation. Define $M = S \circ T^{-1}$. Then $M(Tz_2) = S(z_2) = 1$, $M(Tz_3) = S(z_3) = 0$, and $M(Tz_3) = S(z_3) = \infty$. Hence $M(z) = S \circ T^{-1}(z) = (z, Tz_2, Tz_3, Tz_4)$. 
Proposition III.3.8. If \( z_2, z_3, z_4 \in \mathbb{C}_\infty \) are distinct, and \( T \) is a Möbius transformation then \((z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)\) for any \( z_1 \in \mathbb{C}_\infty \).

**Proof.** Let \( S(z) = (z, z_1, z_2, z_3, z_4) \) (as defined above). Then \( S \) is a Möbius transformation. Define \( M = S \circ T^{-1} \). Then \( M(Tz_2) = S(z_2) = 1 \), \( M(Tz_3) = S(z_3) = 0 \), and \( M(Tz_3) = S(z_3) = \infty \). Hence \( M(z) = S \circ T^{-1}(z) = (z, Tz_2, Tz_3, Tz_4) \). With \( z = Tz_1 \), we have

\[
S \circ T^{-1}(Tz_1) = (Tz_1, Tz_2, Tz_3, Tz_4),
\]
or

\[
S(z_1) = (z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).
\]
Proposition III.3.8. If \( z_2, z_3, z_4 \in \mathbb{C}_\infty \) are distinct, and \( T \) is a Möbius transformation then \((z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)\) for any \( z_1 \in \mathbb{C}_\infty \).

**Proof.** Let \( S(z) = (z, z_1, z_2, z_3, z_4) \) (as defined above). Then \( S \) is a Möbius transformation. Define \( M = S \circ T^{-1} \). Then \( M(Tz_2) = S(z_2) = 1 \), \( M(Tz_3) = S(z_3) = 0 \), and \( M(Tz_3) = S(z_3) = \infty \). Hence \( M(z) = S \circ T^{-1}(z) = (z, Tz_2, Tz_3, Tz_4) \). With \( z = Tz_1 \), we have

\[
S \circ T^{-1}(Tz_1) = (Tz_1, Tz_2, Tz_3, Tz_4),
\]

or

\[
S(z_1) = (z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).
\]

\(\square\)
Proposition III.3.9

**Proposition III.3.9.** If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct and $\omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty$ are distinct, then there is one and only one Möbius transformation such that $S(z_2) = \omega_2$, $S(z_3) = \omega_3$, and $S(z_4) = \omega_4$.

**Proof.** Define $Tz = (z, z_2, z_3, z_4)$ and $Mz = (z, \omega_2, \omega_3, \omega_4)$. Let $S = M^{-1} \circ T$. 
Proposition III.3.9. If \( z_2, z_3, z_4 \in \mathbb{C}_\infty \) are distinct and \( \omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty \) are distinct, then there is one and only one Möbius transformation such that \( S(z_2) = \omega_2, \ S(z_3) = \omega_3, \) and \( S(z_4) = \omega_4. \)

**Proof.** Define \( Tz = (z, z_2, z_3, z_4) \) and \( Mz = (z, \omega_2, \omega_3, \omega_4). \) Let \( S = M^{-1} \circ T. \) Then

\[
S(z_2) = M^{-1} \circ T(z_2) = M^{-1}(1) = \omega_2, \\
S(z_3) = M^{-1} \circ T(z_3) = M^{-1}(0) = \omega_3, \text{ and} \\
S(z_4) = M^{-1} \circ T(z_4) = M^{-1}(\infty) = \omega_4.
\]
Proposition III.3.9. If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct and $\omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty$ are distinct, then there is one and only one Möbius transformation such that $S(z_2) = \omega_2$, $S(z_3) = \omega_3$, and $S(z_4) = \omega_4$.

Proof. Define $Tz = (z, z_2, z_3, z_4)$ and $Mz = (z, \omega_2, \omega_3, \omega_4)$. Let $S = M^{-1} \circ T$. Then

$$
S(z_2) = M^{-1} \circ T(z_2) = M^{-1}(1) = \omega_2,
$$

$$
S(z_3) = M^{-1} \circ T(z_3) = M^{-1}(0) = \omega_3, \text{ and}
$$

$$
S(z_4) = M^{-1} \circ T(z_4) = M^{-1}(\infty) = \omega_4.
$$

If $R$ is another Möbius transformation with $Rz_i = \omega_i$ for $i = 2, 3, 4$, then $R^{-1} \circ S$ fixed $z_2, z_3, z_4$ and so $R^{-1} \circ S = I$, or $R = S$. So the transformation is unique. \qed
Proposition III.3.9. If \( z_2, z_3, z_4 \in \mathbb{C}_\infty \) are distinct and \( \omega_2, \omega_3, \omega_4 \in \mathbb{C}_\infty \) are distinct, then there is one and only one Möbius transformation such that \( S(z_2) = \omega_2, S(z_3) = \omega_3, \) and \( S(z_4) = \omega_4. \)

**Proof.** Define \( Tz = (z, z_2, z_3, z_4) \) and \( Mz = (z, \omega_2, \omega_3, \omega_4). \) Let \( S = M^{-1} \circ T. \) Then

\[
S(z_2) = M^{-1} \circ T(z_2) = M^{-1}(1) = \omega_2, \\
S(z_3) = M^{-1} \circ T(z_3) = M^{-1}(0) = \omega_3, \text{ and} \\
S(z_4) = M^{-1} \circ T(z_4) = M^{-1}(\infty) = \omega_4.
\]

If \( R \) is another Möbius transformation with \( Rz_i = \omega_i \) for \( i = 2, 3, 4, \) then \( R^{-1} \circ S \) fixed \( z_2, z_3, z_4 \) and so \( R^{-1} \circ S = I, \) or \( R = S. \) So the transformation is unique.
Proposition III.3.10

Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof. Let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined as $S(z) = (z, z_2, z_3, z_4)$. Then $S(z)$ is real if and only if $(z, z_2, z_3, z_4)$ is real. So

$$\{z \mid (z, z_2, z_3, z_4) \in \mathbb{R}\} = \{z \mid S(z) \in \mathbb{R}\} = \{z \mid z \in S^{-1}(\mathbb{R})\}.$$  

So we show that the inverse image of $\mathbb{R}_\infty$ is a circle/cline under any Möbius transformation. Let $S(z) = \frac{az + b}{cz + d}$. If $z = x \in \mathbb{R}$ and if $\omega = S^{-1} \neq \infty$ (so $x \neq -d/c$) then $x = S(\omega) \in \mathbb{R}$ and so $S(\omega) = \overline{S(\omega)}$. So $\frac{a\omega + b}{c\omega + d} = \frac{\overline{a\omega + b}}{\overline{c\omega + d}}$. Therefore $(a\omega + b)(\overline{c\omega + d}) = (c\omega + d)(\overline{a\omega + b})$ or $a\overline{c}|\omega|^2 + a\overline{d}\omega + b\overline{c}\omega + b\overline{d} = \overline{ac}|\omega|^2 + d\overline{a}\omega + c\overline{b}\omega + d\overline{b}$ or

$$(a\overline{c} - \overline{ac})|\omega|^2 + (a\overline{d} - c\overline{b})\omega + (b\overline{c} - d\overline{a})\overline{\omega} + (b\overline{d} - d\overline{b}) = 0. \quad (3.11)$$
Proposition III.3.10

Proposition III.3.10. Let \( z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty \) be distinct. Then the cross ratio \((z_1, z_2, z_3, z_4)\) is real if and only if the four points lie on a circle/cline.

Proof. Let \( S : \mathbb{C}_\infty \to \mathbb{C}_\infty \) be defined as \( S(z) = (z, z_2, z_3, z_4) \). Then \( S(z) \) is real if and only if \((z, z_2, z_3, z_4)\) is real. So

\[
\{ z \mid (z, z_2, z_3, z_4) \in \mathbb{R} \} = \{ z \mid S(z) \in \mathbb{R} \} = \{ z \mid z \in S^{-1}(\mathbb{R}) \}.
\]

So we show that the inverse image of \( \mathbb{R}_\infty \) is a circle/cline under any Möbius transformation. Let \( S(z) = \frac{az+b}{cz+d} \). If \( z = x \in \mathbb{R} \) and if \( \omega = S^{-1} \neq \infty \) (so \( x \neq -d/c \)) then \( x = S(\omega) \in \mathbb{R} \) and so \( S(\omega) = \bar{S}(\omega) \).

So \( \frac{a\omega+b}{c\omega+d} = \frac{\bar{a}\omega+\bar{b}}{\bar{c}\omega+d} \). Therefore \((a\omega+b)(\bar{c}\omega+\bar{d}) = (c\omega+d)(\bar{a}\omega+\bar{b})\) or \(a\bar{c}|\omega|^2 + a\bar{d}\omega + b\bar{c}\omega + b\bar{d} = \bar{a}c|\omega|^2 + d\bar{a}\omega + c\bar{b}\omega + d\bar{b}\) or

\[
(a\bar{c} - \bar{a}c)|\omega|^2 + (a\bar{d} - c\bar{b})\omega + (b\bar{c} - d\bar{a})\bar{\omega} + (b\bar{d} - d\bar{b}) = 0. \tag{3.11}
\]
Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof (continued).

Case 1. Suppose $a\bar{c}$ is real.
Proposition III.3.10. \( \text{Let } z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty \text{ be distinct. Then the cross ratio } (z_1, z_2, z_3, z_4) \text{ is real if and only if the four points lie on a circle/cline.} \)

Proof (continued).

Case 1. Suppose \( a \bar{c} \text{ is real.} \) Then \( a \bar{c} - \bar{a}c = 0. \) Let \( \alpha = 2(\bar{a}d - c\bar{b}) \) and \( \beta = i(\bar{b}d - d\bar{b}) \).
Proposition III.3.10. Let \( z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty \) be distinct. Then the cross ratio \((z_1, z_2, z_3, z_4)\) is real if and only if the four points lie on a circle/cline.

Proof (continued).

Case 1. Suppose \( a \bar{c} \) is real. Then \( a \bar{c} - \bar{a}c = 0 \). Let \( \alpha = 2(a \bar{d} - c \bar{b}) \) and \( \beta = i(b \bar{d} - d \bar{b}) \). Equation (3.11) then becomes

\[
\frac{\alpha}{2} \omega - \frac{\bar{\alpha}}{2} \bar{\omega} - i\beta = 0 \quad \text{or} \quad \operatorname{Im}(\alpha \omega) - i\beta = 0 \quad \text{or} \quad \operatorname{Im}(\alpha \omega - \beta) = 0,
\]

since

\[
\beta = i(b \bar{d} - d \bar{b}) = i(2\operatorname{Im}(b \bar{d})) = -2\operatorname{Im}(b \bar{d}) \in \mathbb{R}.
\]
Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof (continued).

Case 1. Suppose $a\overline{c}$ is real. Then $a\overline{c} - \overline{a}c = 0$. Let $\alpha = 2(a\overline{d} - c\overline{b})$ and $\beta = i(b\overline{d} - d\overline{b})$. Equation (3.11) then becomes $\frac{\alpha}{2} \omega - \frac{\overline{\alpha}}{2} \overline{\omega} - i\beta = 0$ or $i\text{Im}(\alpha \omega) - i\beta = 0$ or $\text{Im}(\alpha \omega - \beta) = 0$, since $\beta = i(b\overline{d} - d\overline{b}) = i(i2\text{Im}(b\overline{d})) = -2\text{Im}(b\overline{d}) \in \mathbb{R}$.

Now $\text{Im}(\alpha \omega - \beta) = 0$ for fixed $\alpha, \beta$ implies that all such $\omega$ lie on a line (see Section I.5).
Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof (continued).

Case 1. Suppose $ac$ is real. Then $ac - \overline{ac} = 0$. Let $\alpha = 2(ad - c\overline{b})$ and $\beta = i(b\overline{d} - d\overline{b})$. Equation (3.11) then becomes $\frac{\alpha}{2} \omega - \overline{\frac{\alpha}{2} \omega} - i\beta = 0$ or $i\text{Im}(\alpha \omega) - i\beta = 0$ or $\text{Im}(\alpha \omega - \beta) = 0$, since

$$\beta = i(b\overline{d} - d\overline{b}) = i(2\text{Im}(b\overline{d}) = -2\text{Im}(b\overline{d}) \in \mathbb{R}.$$ 

Now $\text{Im}(\alpha \omega - \beta) = 0$ for fixed $\alpha, \beta$ implies that all such $\omega$ lie on a line (see Section I.5).
Case 2. Suppose $a\bar{c}$ is not real.

Then equation (3.11) becomes

$$|\omega|^2 + \gamma \omega + \gamma \bar{\omega} - \delta = 0$$

where

$$\gamma = \frac{b}{ac} - \frac{d}{a}$$

and

$$\delta = \frac{bd}{ac} - \frac{b}{a}d$$

$$= i \frac{2 \text{Im}(bd)}{i \frac{2 \text{Im}(ac)}} \in \mathbb{R}.$$
Proof (continued).
Case 2. Suppose $a\bar{c}$ is not real. Then equation (3.11) becomes

$$|\omega|^2 + \gamma\omega + \bar{\gamma}\omega - \delta = 0$$

where $\gamma = \frac{b\bar{c} - d\bar{a}}{a\bar{c} - \bar{a}c}$ and

$$\delta = \frac{\bar{b}d - bd}{a\bar{c} - \bar{a}c} = \frac{i2\text{Im}(\bar{b}d)}{i2\text{Im}(a\bar{c})} = \frac{\text{Im}(\bar{b}d)}{\text{Im}(a\bar{c})} \in \mathbb{R}.$$
Proposition III.3.10 (continued 2)

Proof (continued).
Case 2. Suppose $ac$ is not real. Then equation (3.11) becomes

$$|\omega|^2 + \overline{\gamma}\omega + \gamma\overline{\omega} - \delta = 0$$

where $\gamma = \frac{b\overline{c} - d\overline{a}}{a\overline{c} - \overline{ac}}$ and

$$\delta = \frac{bd - bd}{a\overline{c} - \overline{ac}} = \frac{i2\text{Im}(bd)}{i2\text{Im}(ac)} = \frac{\text{Im}(bd)}{\text{Im}(ac)} \in \mathbb{R}.$$  So

$$|\omega|^2 + \overline{\gamma}\omega + \gamma\overline{\omega} + |\gamma|^2 = |\gamma|^2 + \delta,$$

or

$$|\omega + \gamma|^2 = (\omega + \gamma)(\overline{\omega} + \overline{\gamma}) = |\gamma|^2 + \delta.$$
Proposition III.3.10 (continued 2)

Proof (continued).

Case 2. Suppose \( a\bar{c} \) is not real. Then equation (3.11) becomes

\[
|\omega|^2 + \gamma \omega + \gamma \bar{\omega} - \delta = 0\quad \text{where } \gamma = \frac{b\bar{c} - d\bar{a}}{a\bar{c} - \bar{a}c} \quad \text{and}
\]

\[
\delta = \frac{bd - b\bar{d}}{a\bar{c} - \bar{a}c} = \frac{i2\text{Im}(bd)}{i2\text{Im}(a\bar{c})} = \frac{\text{Im}(bd)}{\text{Im}(a\bar{c})} \in \mathbb{R}. \quad \text{So}
\]

\[
|\omega|^2 + \gamma \omega + \gamma \bar{\omega} + |\gamma|^2 = |\gamma|^2 + \delta, \quad \text{or } |\omega + \gamma|^2 = (\omega + \gamma)(\bar{\omega} + \bar{\gamma}) = |\gamma|^2 + \delta.
\]

Hence

\[
|\omega + \gamma| = \sqrt{|\gamma|^2 + \delta} = \sqrt{\frac{b\bar{c} - d\bar{a}}{a\bar{c} - \bar{a}c} \frac{bc - \bar{d}a}{\bar{a} - a\bar{c}} + \frac{bd - b\bar{d}}{a\bar{c} - \bar{a}c} \frac{ac - a\bar{c}}{\delta \bar{a}c - a\bar{c}}} = \frac{1}{|a\bar{c} - \bar{a}c|}\{b\bar{c}bc - b\bar{c}d\bar{a} - d\bar{a}bc + d\bar{a}d\bar{a} + b\bar{d}ac - bda\bar{c} - b\bar{d}a\bar{c} + b\bar{d}ac\}^{1/2}
\]
Proof (continued).

Case 2. Suppose $a\bar{c}$ is not real. Then equation (3.11) becomes

$$|\omega|^2 + \bar{\gamma}\omega + \gamma\bar{\omega} - \delta = 0$$

where $\gamma = \frac{b\bar{c} - d\bar{a}}{a\bar{c} - \bar{a}c}$ and

$$\delta = \frac{\bar{b}d - b\bar{d}}{a\bar{c} - \bar{a}c} = \frac{i2\text{Im}(\bar{b}d)}{i2\text{Im}(a\bar{c})} = \frac{\text{Im}(\bar{b}d)}{\text{Im}(a\bar{c})} \in \mathbb{R}.$$ So

$$|\omega|^2 + \bar{\gamma}\omega + \gamma\bar{\omega} + |\gamma|^2 = |\gamma|^2 + \delta,$$

or $|\omega + \gamma|^2 = (\omega + \gamma)(\bar{\omega} + \bar{\gamma}) = |\gamma|^2 + \delta.$

Hence

$$|\omega + \gamma| = \sqrt{|\gamma|^2 + \delta} = \sqrt{\frac{b\bar{c} - d\bar{a}}{a\bar{c} - \bar{a}c} \frac{\bar{b}c - \bar{d}a}{\bar{a} - a\bar{c}} + \frac{\bar{b}d - b\bar{d}}{a\bar{c} - \bar{a}c} \frac{\bar{a}c - a\bar{c}}{\bar{a} - a\bar{c}}}$$

$$= \frac{1}{|a\bar{c} - \bar{a}c|} \{b\bar{c}b\bar{c} - b\bar{c}d\bar{a} - d\bar{a}b\bar{c} + d\bar{a}d\bar{a} + \bar{b}d\bar{a}\bar{c} - \bar{b}da\bar{c} - b\bar{d}\bar{a}c + b\bar{d}\bar{a}c\}^{1/2}.$$
Proposition III.3.10 (continued 3)

**Proposition III.3.10.** Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof (continued).

\[
\begin{align*}
&= \frac{1}{|ac - \bar{ac}|} \left\{ bc\bar{bc} - b\bar{c}d a - d\bar{a}bc + d\bar{a}da + \bar{b}d\bar{a}c - \bar{b}da\bar{c} - b\bar{d}\bar{a}c + b\bar{d}a\bar{c} \right\}^{1/2} \\
&= \frac{1}{|ac - \bar{ac}|} \left\{ \bar{b}\bar{c}(bc - ad) - \bar{a}d(-ad + bc) \right\}^{1/2} \\
&= \frac{1}{|ac - \bar{ac}|} \sqrt{(\bar{b}\bar{c} - \bar{a}d)(bc - ad)} = \frac{|ad - bc|}{|ac - \bar{ac}|} > 0
\end{align*}
\]

since $ad - bc \neq 0$. 


Proposition III.3.10. Let \( z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty \) be distinct. Then the cross ratio \((z_1, z_2, z_3, z_4)\) is real if and only if the four points lie on a circle/cline.

Proof (continued).

\[
= \frac{1}{|a\bar{c} - \bar{ac}|} \left\{ bc \bar{b}c - b\bar{c}da - d\bar{a}bc + d\bar{a}da + \bar{b}d\bar{a}c - \bar{bd}ac - b\bar{d}\bar{a}c + \bar{bd}a\bar{c} \right\}^{1/2}
\]

\[
= \frac{1}{|a\bar{c} - \bar{ac}|} \left\{ \bar{b}c(bc - ad) - \bar{a}d(-ad + bc) \right\}^{1/2}
\]

\[
= \frac{1}{|a\bar{c} - \bar{ac}|} \sqrt{(\bar{b}c - \bar{a}d)(bc - ad)} = \frac{|ad - bc|}{|a\bar{c} - \bar{ac}|} > 0
\]

since \( ad - bc \neq 0 \). So \( \omega \) lies on a circle of center \(-\delta\) with radius \( \frac{|ad - bc|}{|a\bar{c} - \bar{ac}|} \), and the result follows.
Proposition III.3.10. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Then the cross ratio $(z_1, z_2, z_3, z_4)$ is real if and only if the four points lie on a circle/cline.

Proof (continued).

\[
= \frac{1}{|ac - \bar{ac}|} \left\{ bc\bar{bc} - b\bar{c}da - d\bar{a}bc + d\bar{a}da + b\bar{d}\bar{ac} - bda\bar{c} - b\bar{d}\bar{ac} + b\bar{d}a\bar{c} \right\}^{1/2}
\]

\[
= \frac{1}{|ac - \bar{ac}|} \left\{ \bar{b}\bar{c}(bc - ad) - \bar{a}d(-ad + bc) \right\}^{1/2}
\]

\[
= \frac{1}{|ac - \bar{ac}|} \sqrt{(\bar{b}\bar{c} - \bar{a}d)(bc - ad)} = \frac{|ad - bc|}{|ac - \bar{ac}|} > 0
\]

since $ad - bc \neq 0$. So $\omega$ lies on a circle of center $-\delta$ with radius $\frac{|ad - bc|}{|ac - \bar{ac}|}$, and the result follows.

Proof. Let $\Gamma$ be a circle/cline in $\mathbb{C}_\infty$ and let $S$ be a Möbius transformation. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points on $\Gamma$. Define $\omega_j = S(z_j)$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle/cline $\Gamma'$. Since $S$ is invertible and one to one, $\omega_2, \omega_3, \omega_4$ are distinct. By Proposition III.3.8, $(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4) = (Sz, \omega_2, \omega_3, \omega_4)$ for each $a \in \mathbb{C}_\infty$. Now for each $z \in \Gamma$, $(z, z_2, z_3, z_4)$ is real by Proposition III.3.10. So $(Sz, \omega_2, \omega_3, \omega_4)$ is real and again by Proposition III.3.10, $Sz$ lies on on $\Gamma'$, the circle/cline containing $\omega_2, \omega_3, \omega_4$. So $S(\Gamma) = \Gamma'$ (recall that $S$ maps $\mathbb{C}_\infty$ one to one and onto $\mathbb{C}_\infty$).

Proof. Let $\Gamma$ be a circle/cline in $\mathbb{C}_\infty$ and let $S$ be a Möbius transformation. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points on $\Gamma$. Define $\omega_j = S(z_j)$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle/cline $\Gamma'$ ($S$ is invertible and so one to one, so $\omega_2, \omega_3, \omega_4$ are distinct).
**Theorem III.3.14**. A Möbius transformation takes circles/clines onto circles/clines.

**Proof.** Let $\Gamma$ be a circle/cline in $\mathbb{C}_\infty$ and let $S$ be a Möbius transformation. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points on $\Gamma$. Define $\omega_j = S(z_j)$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle/cline $\Gamma'$ ($S$ is invertible and so one to one, so $\omega_2, \omega_3, \omega_4$ are distinct). By Proposition III.3.8,

$$(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4) = (Sz, \omega_2, \omega_3, \omega_4)$$

for each $a \in \mathbb{C}_\infty$. Now for each $z \in \Gamma$, $(z, z_2, z_3, z_4)$ is real by Proposition III.3.10.

Proof. Let $\Gamma$ be a circle/cline in $\mathbb{C}_\infty$ and let $S$ be a Möbius transformation. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points on $\Gamma$. Define $\omega_j = S(z_j)$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle/cline $\Gamma'$ ($S$ is invertible and so one to one, so $\omega_2, \omega_3, \omega_4$ are distinct). By Proposition III.3.8,

$$(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4) = (Sz, \omega_2, \omega_3, \omega_4)$$

for each $a \in \mathbb{C}_\infty$. Now for each $z \in \Gamma$, $(z, z_2, z_3, z_4)$ is real by Proposition III.3.10. So $(Sz, \omega_2, \omega_3, \omega_4)$ is real and again by Proposition III.3.10, $Sz$ lies on on $\Gamma'$, the circle/cline containing $\omega_2, \omega_3, \omega_4$. So $S(\Gamma) = \Gamma'$ (recall that $S$ maps $\mathbb{C}_\infty$ one to one and onto $\mathbb{C}_\infty$).

**Proof.** Let $\Gamma$ be a circle/cline in $\mathbb{C}_\infty$ and let $S$ be a Möbius transformation. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points on $\Gamma$. Define $\omega_j = S(z_j)$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle/cline $\Gamma'$ ($S$ is invertible and so one to one, so $\omega_2, \omega_3, \omega_4$ are distinct). By Proposition III.3.8,

$$(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4) = (Sz, \omega_2, \omega_3, \omega_4)$$

for each $a \in \mathbb{C}_\infty$. Now for each $z \in \Gamma$, $(z, z_2, z_3, z_4)$ is real by Proposition III.3.10. So $(Sz, \omega_2, \omega_3, \omega_4)$ is real and again by Proposition III.3.10, $Sz$ lies on on $\Gamma'$, the circle/cline containing $\omega_2, \omega_3, \omega_4$. So $S(\Gamma) = \Gamma'$ (recall that $S$ maps $\mathbb{C}_\infty$ one to one and onto $\mathbb{C}_\infty$).
Theorem III.3.19. Symmetry Principle

If a Möbius transformation takes a circle/cline $\Gamma_1$ onto the circle/cline $\Gamma_2$ then any pair of points symmetric with respect to $\Gamma_1$ are mapped by $T$ onto a pair of points symmetric with respect to $\Gamma_2$.

Proof. Let $z_2, z_3, z_4 \in \Gamma_1$ be distinct. Let $z$ and $z^*$ be symmetric with respect to $\Gamma_1$. 
If a Möbius transformation takes a circle/cline $\Gamma_1$ onto the circle/cline $\Gamma_2$ then any pair of points symmetric with respect to $\Gamma_1$ are mapped by $T$ onto a pair of points symmetric with respect to $\Gamma_2$.

Proof. Let $z_2, z_3, z_4 \in \Gamma_1$ be distinct. Let $z$ and $z^*$ be symmetric with respect to $\Gamma_1$. Then

\[
(Tz^*, Tz_2, Tz_3, Tz_4) = (z^*, z_2, z_3, z_4) \quad \text{by Proposition III.3.8}
\]
\[
= (z, z_2, z_3, z_4) \quad \text{since $z$ and $z^*$ are symmetric wrt $\Gamma$}
\]
\[
= (Tz, Tz_2, Tz_3, Tz_4) \quad \text{by Proposition III.3.8}.
\]
If a Möbius transformation takes a circle/cline $\Gamma_1$ onto the circle/cline $\Gamma_2$ then any pair of points symmetric with respect to $\Gamma_1$ are mapped by $T$ onto a pair of points symmetric with respect to $\Gamma_2$.

Proof. Let $z_2, z_3, z_4 \in \Gamma_1$ be distinct. Let $z$ and $z^*$ be symmetric with respect to $\Gamma_1$. Then

\[
(Tz^*, Tz_2, Tz_3, Tz_4) = (z^*, z_2, z_3, z_4) \text{ by Proposition III.3.8}
\]

\[
= (z, z_2, z_3, z_4) \text{ since } z \text{ and } z^* \text{ are symmetric wrt } \Gamma
\]

\[
= (Tz, Tz_2, Tz_3, Tz_4) \text{ by Proposition III.3.8.}
\]

So $Tz^*$ and $Tz$ are symmetric with respect to $\Gamma_2 = T(\Gamma_1)$. 

\hfill \Box

If a Möbius transformation takes a circle/cline $\Gamma_1$ onto the circle/cline $\Gamma_2$ then any pair of points symmetric with respect to $\Gamma_1$ are mapped by $T$ onto a pair of points symmetric with respect to $\Gamma_2$.

Proof. Let $z_2, z_3, z_4 \in \Gamma_1$ be distinct. Let $z$ and $z^*$ be symmetric with respect to $\Gamma_1$. Then

$$(Tz^*, Tz_2, Tz_3, Tz_4) = (z^*, z_2, z_3, z_4)$$

by Proposition III.3.8

$$= (z, z_2, z_3, z_4)$$

since $z$ and $z^*$ are symmetric wrt $\Gamma$

$$= (Tz, Tz_2, Tz_3, Tz_4)$$

by Proposition III.3.8.

So $Tz^*$ and $Tz$ are symmetric with respect to $\Gamma_2 = T(\Gamma_1)$. \qed