Chapter IV. Complex Integration
IV.1. Riemann-Stieltjes Integrals—Proofs
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$$V(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$ 

Proof. Assume that $\gamma$ is smooth (the case of piecewise smooth following by summing). Let $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$. 
Proposition IV.1.3. If \( \gamma : [a, b] \rightarrow \mathbb{C} \) is piecewise smooth then \( \gamma \) is of bounded variation and

\[
V(\gamma) = \int_{a}^{b} |\gamma'(t)| \, dt.
\]

Proof. Assume that \( \gamma \) is smooth (the case of piecewise smooth following by summing). Let \( P = \{ a = t_0 < t_1 < \cdots < t_m = b \} \). Then

\[
V(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})|
\]

\[
= \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right| 
\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} |\gamma'(t)| \, dt = \int_{a}^{b} |\gamma'(t)| \, dt.
\]
Proposition IV.1.3. If \( \gamma : [a, b] \to \mathbb{C} \) is piecewise smooth then \( \gamma \) is of bounded variation and

\[
V(\gamma) = \int_a^b |\gamma'(t)| \, dt.
\]

Proof. Assume that \( \gamma \) is smooth (the case of piecewise smooth following by summing). Let \( P = \{a = t_0 < t_1 < \cdots < t_m = b\} \). Then

\[
V(\gamma; P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|
\]

\[
= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right| \text{ by the FTC since } \gamma \text{ is smooth}
\]

\[
\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| \, dt = \int_a^b |\gamma'(t)| \, dt.
\]

Hence \( \gamma \) is of bounded variation since

\[
V(\gamma) \leq \int_a^b |\gamma'(t)| \, dt, \quad (\ast)
\]
Theorem IV.1.3

**Proposition IV.1.3.** If \( \gamma : [a, b] \to \mathbb{C} \) is piecewise smooth then \( \gamma \) is of bounded variation and

\[
V(\gamma) = \int_a^b |\gamma'(t)| \, dt.
\]

**Proof.** Assume that \( \gamma \) is smooth (the case of piecewise smooth following by summing). Let \( P = \{a = t_0 < t_1 < \cdots < t_m = b\} \). Then

\[
V(\gamma; P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|
\]

\[
= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right| \text{ by the FTC since } \gamma \text{ is smooth}
\]

\[
\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| \, dt = \int_a^b |\gamma'(t)| \, dt.
\]

Hence \( \gamma \) is of bounded variation since \( V(\gamma) \leq \int_a^b |\gamma'(t)| \, dt, \quad (*) \)
Theorem IV.1.3 (continued 1)

Proof (continued). Since \( \gamma' \) is continuous and \([a, b]\) is compact, then \( \gamma' \) is uniformly continuous. So if \( \varepsilon > 0 \), there exists \( \delta_1 > 0 \) such that \( |s - t| < \delta_1 \) implies \( |\gamma'(s) - \gamma'(t)| < \varepsilon \).
Proof (continued). Since $\gamma'$ is continuous and $[a, b]$ is compact, then $\gamma'$ is uniformly continuous. So if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also by definition of integral, there exists $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and $\|P\| = \max\{t_k - t_{k-1} \mid 1 \leq k \leq m\} < \delta_2$ implies

$$\left| \int_a^b |\gamma'(t)| \, dt - \sum_{k=1}^{m} |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon$$

where $\tau_k$ is any point in $[t_{k-1}, t_k]$. 
Proof (continued). Since \( \gamma' \) is continuous and \([a, b]\) is compact, then \( \gamma' \) is uniformly continuous. So if \( \varepsilon > 0 \), there exists \( \delta_1 > 0 \) such that 
\[
|s - t| < \delta_1 \implies |\gamma'(s) - \gamma'(t)| < \varepsilon.
\]
Also by definition of integral, there exists \( \delta_2 > 0 \) such that if \( P = \{a = t_0 < t_1 < \cdots < t_m = b\} \) and
\[
\|P\| = \max\{t_k - t_{k-1} \mid 1 \leq k \leq m\} < \delta_2
\]
implies
\[
\left| \int_a^b |\gamma'(t)| \, dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon
\]
where \( \tau_k \) is any point in \([t_{k-1}, t_k]\). Hence
\[
\int_a^b |\gamma'(t)| \, dt \leq \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1})
\]
\[
= \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) \, dt \right| \quad \text{since } \gamma'(\tau_k) \text{ is constant}.
\]
Proof (continued). Since $\gamma'$ is continuous and $[a, b]$ is compact, then $\gamma'$ is uniformly continuous. So if $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|s - t| < \delta_1$ implies $|\gamma'(s) - \gamma'(t)| < \varepsilon$. Also by definition of integral, there exists $\delta_2 > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ and 

$$
\|P\| = \max\{t_k - t_{k-1} \mid 1 \leq k \leq m\} < \delta_2 \text{ implies }
$$

$$
\left| \int_a^b |\gamma'(t)| \, dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \varepsilon \text{ where } \tau_k \text{ is any point in } [t_{k-1}, t_k].
$$

Hence

$$
\int_a^b |\gamma'(t)| \, dt \leq \varepsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) = \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) \, dt \right| \text{ since } \gamma'(\tau_k) \text{ is constant}
$$
Theorem IV.1.3 (continued 2)

Proof (continued).

\[
\int_a^b |\gamma'(t)| \, dt \leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)] \, dt \right|
\]

\[
\leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)] \, dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right|
\]
Theorem IV.1.3 (continued 2)

Proof (continued).

\[
\int_a^b |\gamma'(t)| \, dt \leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)] \, dt \right|
\]

\[
\leq \varepsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} [\gamma'(\tau_k) - \gamma'(t)] \, dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right|
\]

If \( ||P|| < \delta = \min\{\delta_1, \delta_2\} \) then \( |\gamma'(\tau_k) - \gamma'(t)| < \varepsilon \) for \( t \in [t_{k-1}, t_k] \) and

\[
\int_a^b |\gamma'(t)| \, dt \leq \varepsilon + \varepsilon (b - a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})|
\]

\[
= \varepsilon [1 + (b - a)] + v(\gamma; P) \leq \varepsilon [1 + b - a] + V(\gamma).
\]

Since \( \varepsilon > 0 \) is arbitrary, \( \int_a^b |\gamma'(t)| \, dt \leq V(\gamma) \), and we have equality combining with \((*)\).
Theorem IV.1.3 (continued 2)

Proof (continued).

\[ \int_{a}^{b} |\gamma'(t)| \, dt \leq \varepsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_{k}} [\gamma'(\tau_k) - \gamma'(t) + \gamma'(t)] \, dt \right| \]

\[ \leq \varepsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_{k}} [\gamma'(\tau_k) - \gamma'(t)] \, dt \right| + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_{k}} \gamma'(t) \, dt \right|. \]

If \( \|P\| < \delta = \min\{\delta_1, \delta_2\} \) then \( |\gamma'(\tau_k) - \gamma'(t)| < \varepsilon \) for \( t \in [t_{k-1}, t_k] \) and

\[ \int_{a}^{b} |\gamma'(t)| \, dt \leq \varepsilon + \varepsilon (b - a) + \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \]

\[ = \varepsilon [1 + (b - a)] + v(\gamma; P) \leq \varepsilon [1 + b - a] + V(\gamma). \]

Since \( \varepsilon > 0 \) is arbitrary, \( \int_{a}^{b} |\gamma'(t)| \, dt \leq V(\gamma) \), and we have equality combining with (*)
Theorem IV.1.4

**Theorem IV.1.4.** Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then there is a complex number $I$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max\{t_k - t_{k-1}\} < \delta$, then

$$\left| I - \sum_{k=1}^{m} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon$$

for whatever choice of points $\tau_k$, where $\tau_k \in [t_{k-1}, t_k]$. The number $I$ is called the **Riemann-Stieltjes integral** of $f$ with respect to $\gamma$ over $[a, b]$, denoted

$$I = \int_{a}^{b} f \, d\gamma = \int_{a}^{b} f(t) \, d\gamma(t).$$
Theorem IV.1.4 (proof 1)

**Proof.** Since \( f \) is continuous and \([a, b]\) is compact, then \( f \) is uniformly continuous on \([a, b]\). So for all \( \varepsilon = 1/m \ (m \in \mathbb{N}) \) there exists \( \delta_m > 0 \) (where we take \( \delta_1 > \delta_2 > \delta_3 > \cdots \)) such that if \(|s - t| < \delta_m\) then \(|f(s) - f(t)| < 1/m\). For each \( m \in \mathbb{N} \), let \( \mathcal{P}_m \) be the set of all partitions \( P \) of \([a, b]\) such that \(|P| < \delta_m\). So \( \mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots \).
Theorem IV.1.4 (proof 1)

**Proof.** Since $f$ is continuous and $[a, b]$ is compact, then $f$ is uniformly continuous on $[a, b]$. So for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) there exists $\delta_m > 0$ (where we take $\delta_1 > \delta_2 > \delta_3 > \cdots$) such that if $|s - t| < \delta_m$ then $|f(s) - f(t)| < 1/m$. For each $m \in \mathbb{N}$, let $\mathcal{P}_m$ be the set of all partitions $P$ of $[a, b]$ such that $\|P\| < \delta_m$. So $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots$. Define $F_m$ (for each $m \in \mathbb{N}$) as the closure of the set:

$$\left\{ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \mid P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}. \quad (*)$$

We now show that the diameter of set $(*)$ is $\leq 2/mV(\gamma)$ for each $m \in \mathbb{N}$ for each $m \in \mathbb{N}$. 
**Theorem IV.1.4 (proof 1)**

**Proof.** Since $f$ is continuous and $[a, b]$ is compact, then $f$ is uniformly continuous on $[a, b]$. So for all $\varepsilon = 1/m$ ($m \in \mathbb{N}$) there exists $\delta_m > 0$ (where we take $\delta_1 > \delta_2 > \delta_3 > \cdots$) such that if $|s - t| < \delta_m$ then $|f(s) - f(t)| < 1/m$. For each $m \in \mathbb{N}$, let $\mathcal{P}_m$ be the set of all partitions $P$ of $[a, b]$ such that $\|P\| < \delta_m$. So $\mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots$. Define $F_m$ (for each $m \in \mathbb{N}$) as the closure of the set:

$$
\left\{ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \middle| P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}.
$$

We now show that the diameter of set $(\ast)$ is $\leq 2/mV(\gamma)$ for each $m \in \mathbb{N}$ for each $m \in \mathbb{N}$. If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$, then denote by $S(P)$ a sum of the form $\sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})]$ where $\tau_k$ is any point with $t_{k-1} \leq \tau_k \leq t_k$. Fix $m \in \mathbb{N}$ and let $P \in \mathcal{P}_m$. 
Theorem IV.1.4 (proof 1)

Proof. Since \( f \) is continuous and \([a, b]\) is compact, then \( f \) is uniformly continuous on \([a, b]\). So for all \( \varepsilon = 1/m \) (\( m \in \mathbb{N} \)) there exists \( \delta_m > 0 \) (where we take \( \delta_1 > \delta_2 > \delta_3 > \cdots \)) such that if \(|s - t| < \delta_m\) then \(|f(s) - f(t)| < 1/m\). For each \( m \in \mathbb{N} \), let \( \mathcal{P}_m \) be the set of all partitions \( P \) of \([a, b]\) such that \(||P|| < \delta_m\). So \( \mathcal{P}_1 \supset \mathcal{P}_2 \supset \mathcal{P}_3 \supset \cdots \). Define \( F_m \) (for each \( m \in \mathbb{N} \)) as the closure of the set:

\[
\left\{ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \middle| P \in \mathcal{P}_m \text{ and } \tau_k \in (t_{k-1}, t_k) \right\}.
\]  

We now show that the diameter of set (\( \ast \)) is \( \leq 2/mV(\gamma) \) for each \( m \in \mathbb{N} \) for each \( m \in \mathbb{N} \). If \( P = \{t_0 < t_1 < \cdots < t_n\} \) is a partition of \([a, b]\), then denote by \( S(P) \) a sum of the form \( \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \) where \( \tau_k \) is any point with \( t_{k-1} \leq \tau_k \leq t_k \). Fix \( m \in \mathbb{N} \) and let \( P \in \mathcal{P}_m \).
Proof (continued). (1) Suppose $P \subset Q$ (and so $Q \in \mathcal{P}_m$) such that $Q = P \cup \{t^*\}$ where $t_{p-1} < t^* < t_p$ (so $Q$ contains one more point than $P$ and is a refinement of $P$). If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$ and if

$$S(Q) = \sum_{k \neq p} f(\sigma_k)[\gamma(t_k) - \gamma(t_{k-1})] + f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] + f(\sigma')[\gamma(t_p) - \gamma(t^*)]$$

then

$$|S(P) - S(Q)| = \left| \sum_{k \neq p} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] - S(Q) \right|$$

$$= \left| \sum_{k \neq p} (f(\tau_k) - f(\sigma_k))[\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] - f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma')[\gamma(t_p) - \gamma(t^*)] \right|$$
Theorem IV.1.4 (proof 1, continued)

Proof (continued). (1) Suppose $P \subset Q$ (and so $Q \in \mathcal{P}_m$) such that $Q = P \cup \{t^*\}$ where $t_{p-1} < t^* < t_p$ (so $Q$ contains one more point than $P$ and is a refinement of $P$). If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$ and if

$$S(Q) = \sum_{k \neq p} f(\sigma_k)[\gamma(t_k) - \gamma(t_{k-1})] + f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})]$$

$$+ f(\sigma')[\gamma(t_p) - \gamma(t^*)]$$

then

$$|S(P) - S(Q)| = \left| \sum_{k \neq p} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right.$$  

$$+ f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] - S(Q)|$$

$$= \left| \sum_{k \neq p} (f(\tau_k) - f(\sigma_k))[\gamma(t_k) - \gamma(t_{k-1})] + f(\tau_p)[\gamma(t_p) - \gamma(t_{p-1})] 

- f(\sigma)[\gamma(t^*) - \gamma(t_{p-1})] - f(\sigma')[\gamma(t_p) - \gamma(t^*)] \right|$$
Theorem IV.1.4 (proof 1, continued again)

Proof (continued).

\[
\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + |[f(\tau_p) - f(\sigma)][\gamma(t^*) - \gamma(t_{p-1})] + [f(\tau_p) - f(\sigma')][\gamma(t_p) - \gamma(t^*)]| \quad \text{(since } |\tau_k - \sigma_k| < \delta_m \text{ and so } f(\tau_k) - f(\sigma_k)| < 1/m) \\
\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| + \frac{1}{m} |\gamma(t_p) - \gamma(t^*)| \\
\leq \frac{1}{m} V(\gamma) \quad \text{(since } t^* - t_{p-1} < \delta_m \text{ and } |t_p - t^*| < \delta_m)\).
\]

Now if \( P \subset Q \) and \( Q \) contains several more points than \( P \), then the proof follows similarly.
Theorem IV.1.4 (proof 2)

**Proof.** Now let $P$ and $R$ be any two partitions in $\mathcal{P}_m$. Then $Q = P \cup R$ is a refinement of both $P$ and $R$. By the above argument,

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

Therefore, the modulus of the difference of any two elements of set (*) is $\leq \frac{1}{m} V(\gamma)$. That is, the diameter of set (*) is $\leq \frac{2}{m} V(\gamma)$ and so $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$. 
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Therefore, the modulus of the difference of any two elements of set $(\ast)$ is $\leq \frac{1}{m} V(\gamma)$. That is, the diameter of set $(\ast)$ is $\leq \frac{2}{m} V(\gamma)$ and so $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$. So the sets $F_m$ are closed, nested $(F_1 \supset F_2 \supset F_2 \supset \cdots)$, and $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$ (and so $\text{diam}(F_m) \to 0$ as $m \to \infty$).
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Proof. Now let $P$ and $R$ be any two partitions in $\mathcal{P}_m$. Then $Q = P \cup R$ is a refinement of both $P$ and $R$. By the above argument,

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$$\cap_{m=1}^{\infty} F_m = \{I\}$$

for some single $I \in \mathbb{C}$. This value $I$ satisfies the claims of the theorem.
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Proof. Now let $P$ and $R$ be any two partitions in $\mathcal{P}_m$. Then $Q = P \cup R$ is a refinement of both $P$ and $R$. By the above argument,

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

Therefore, the modulus of the difference of any two elements of set $(\ast)$ is $\leq \frac{1}{m} V(\gamma)$. That is, the diameter of set $(\ast)$ is $\leq \frac{2}{m} V(\gamma)$ and so $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$. So the sets $F_m$ are closed, nested $(F_1 \supset F_2 \supset F_2 \supset \cdots)$, and $\text{diam}(F_m) \leq \frac{2}{m} V(\gamma)$ (and so $\text{diam}(F_m) \to 0$ as $m \to \infty$). Therefore by Cantor’s Theorem (Theorem II.3.7),

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Theorem IV.1.9

Theorem IV.1.9. If $\gamma$ is piecewise smooth and $f : [a, b] \rightarrow \mathbb{C}$ is continuous then

$$\int_a^b f \, d\gamma = \int_a^b f(t)\gamma'(t) \, dt.$$ 

Proof. Without loss of generality, $\gamma$ is smooth (the result for piecewise smooth following then from additivity). Also, $\gamma$ can be represented as $\gamma = \gamma_r + i\gamma_i$ where $\gamma_r$ and $\gamma_i$ are real. So also WLOG, $\gamma([a, b]) \subset \mathbb{R}$ (the general result following for complex valued $\gamma$ by linearity).
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$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2} \quad (1.10)$$

and

$$\left| \int_a^b f(t) \gamma'(t) \, dt - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2} \quad (1.11)$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. 


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$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2} \quad (1.10)$$

and

$$\left| \int_a^b f(t)\gamma'(t) \, dt - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \frac{\varepsilon}{2} \quad (1.11)$$

for any choice of $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. 

Complex Analysis

November 24, 2015
Theorem IV.1.9 (continued)

**Theorem IV.1.9.** If $\gamma$ is piecewise smooth and $f : [a, b] \rightarrow \mathbb{C}$ is continuous then

$$\int_{a}^{b} f \, d\gamma = \int_{a}^{b} f(t)\gamma'(t) \, dt.$$ 

**Proof (continued).** By the Mean Value Theorem (for real functions from Calculus 1) there is $\tau_k \in [t_{k-1}, t_k]$ with $\gamma'(\tau_k) = [\gamma(t_k) - \gamma(t_{k-1})]/(t_k - t_{k-1})$.
Theorem IV.1.9. If $\gamma$ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous then
\[ \int_a^b f \, d\gamma = \int_a^b f(t)\gamma'(t) \, dt. \]

Proof (continued). By the Mean Value Theorem (for real functions from Calculus 1) there is $\tau_k \in [t_{k-1}, t_k]$ with $\gamma'(\tau_k) = [\gamma(t_k) - \gamma(t_{k-1})]/(t_k - t_{k-1})$. Thus
\[ \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] = \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}). \]

Therefore
\[ \left| \int_a^b f \, d\gamma - \int_a^b f(t)\gamma'(t) \, dt \right| = \left| \int_a^b f \, d\gamma - \sum_{k=1}^{n} f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| \]
\[ + \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) - \int_a^b f(t)\gamma'(t) \, dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
by (1.10) and (1.11).
Theorem IV.1.9. If $\gamma$ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous then
\[ \int_a^b f \, d\gamma = \int_a^b f(t)\gamma'(t) \, dt. \]

Proof (continued). By the Mean Value Theorem (for real functions from Calculus 1) there is $\tau_k \in [t_{k-1}, t_k]$ with $\gamma'\left(\tau_k\right) = \left[\gamma(t_k) - \gamma(t_{k-1})\right]/(t_k - t_{k-1})$. Thus
\[ \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] = \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}). \]

Therefore
\[ \left| \int_a^b f \, d\gamma - \int_a^b f(t)\gamma'(t) \, dt \right| = \left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)[\gamma(t_k) - \gamma(t_{k-1})] \right| \]
\[ + \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) - \int_a^b f(t)\gamma'(t) \, dt \]< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
by (1.10) and (1.11).
Proposition IV.1.13

Proposition IV.1.13. If \( \gamma : [a, b] \rightarrow \mathbb{C} \) is a rectifiable path and \( \sigma : [c, d] \rightarrow [a, b] \) is a continuous non-decreasing function with \( \sigma(c) = a \) and \( \sigma(d) = b \), then for any \( f \) continuous on \( \{\gamma\} = \{\gamma \circ \sigma\} \) we have

\[
\int_\gamma f = \int_{\gamma \circ \sigma} f.
\]

Proof. Let \( \varepsilon > 0 \) and choose \( \delta_1 > 0 \) such that for \( P_1 = \{c = s_0 < s_1 < \cdots < s_n = d\} \) a partition of \([c, d]\) with \( \|P_1\| < \delta_1 \) and \( s_{k-1} \leq \sigma_k \leq s_k \) we have

\[
\left| \int_{\gamma \circ \sigma} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.
\]
Proposition IV.1.13

**Proposition IV.1.13.** If \( \gamma : [a, b] \rightarrow \mathbb{C} \) is a rectifiable path and \( \sigma : [c, d] \rightarrow [a, b] \) is a continuous non-decreasing function with \( \sigma(c) = a \) and \( \sigma(d) = b \), then for any \( f \) continuous on \( \{ \gamma \} = \{ \gamma \circ \sigma \} \) we have
\[
\int_{\gamma} f = \int_{\gamma \circ \sigma} f.
\]

**Proof.** Let \( \varepsilon > 0 \) and choose \( \delta_1 > 0 \) such that for
\[
P_1 = \{ c = s_0 < s_1 < \cdots < s_n = d \}
\]
a partition of \([c, d]\) with \( \| P_1 \| < \delta_1 \) and \( s_{k-1} \leq \sigma_k \leq s_k \) we have
\[
\left| \int_{\gamma \circ \sigma} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.
\]

Choose \( \delta_2 > 0 \) such that if \( P_2 = \{ a = t_0 < t_1 < \cdots < t_n = b \} \) is a partition of \([a, b]\) with \( \| P_2 \| < \delta_2 \) and \( t_{k-1} < \tau_k < t_k \) then
\[
\left| \int_{\gamma} f - \sum_{k=1}^{n} (\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2}.
\]
Proposition IV.1.13. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have
\[
\int_{\gamma} f = \int_{\gamma \circ \sigma} f.
\]

Proof. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that for $P_1 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ a partition of $[c, d]$ with $\|P_1\| < \delta_1$ and $s_{k-1} \leq \sigma_k \leq s_k$ we have
\[
\left| \int_{\gamma \circ \sigma} f - \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k)) [\gamma \circ \sigma(s_k) - \gamma \circ \sigma(s_{k-1})] \right| < \frac{\varepsilon}{2}.
\]
Choose $\delta_2 > 0$ such that if $P_2 = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P_2\| < \delta_2$ and $t_{k-1} < \tau_k < t_k$ then
\[
\left| \int_{\gamma} f - \sum_{k=1}^{n} (\gamma(\tau_k)) [\gamma(t_k) - \gamma(t_{k-1})] \right| < \frac{\varepsilon}{2}.
\]
Proposition IV.1.13. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have
$$\int_{\gamma} f = \int_{\gamma \circ \sigma} f.$$  

Proof (continued). Since $\phi$ is continuous on $[c, d]$ and $[c, d]$ is compact, then there is a $\delta > 0$ such that $\delta < \delta_1$ and $|\phi(s) - \phi(s')| < \delta_2$ whenever $|s - s'| < \delta$ (by the definition of uniform continuity).
Proposition IV.1.13. If \( \gamma : [a, b] \to \mathbb{C} \) is a rectifiable path and \( \sigma : [c, d] \to [a, b] \) is a continuous non-decreasing function with \( \sigma(c) = a \) and \( \sigma(d) = b \), then for any \( f \) continuous on \( \{ \gamma \} = \{ \gamma \circ \sigma \} \) we have
\[
\int_\gamma f = \int_{\gamma \circ \sigma} f.
\]

Proof (continued). Since \( \varphi \) is continuous on \( [c, d] \) and \( [c, d] \) is compact, then there is a \( \delta > 0 \) such that \( \delta < \delta_1 \) and \( |\varphi(s) - \varphi(s')| < \delta_2 \) whenever \( |s - s'| < \delta \) (by the definition of uniform continuity). So if \( P_2 = \{ c = s_0 < s_1 < \cdots < s_n = d \} \) is a partition of \( [c, d] \) with \( \|P_3\| < \delta < \delta_1 \) and \( t_k = \varphi(s_k) \), then \( P_4 = \{ a = t_0 \leq t_1 \leq \cdots \leq t_n = b \} \) is a partition of \( [a, b] \) with \( \|P_4\| < \delta_2 \).
Proposition IV.1.13. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\sigma : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\sigma(c) = a$ and $\sigma(d) = b$, then for any $f$ continuous on $\{\gamma\} = \{\gamma \circ \sigma\}$ we have $\int_\gamma f = \int_{\gamma \circ \sigma} f$.

Proof (continued). Since $\varphi$ is continuous on $[c, d]$ and $[c, d]$ is compact, then there is a $\delta > 0$ such that $\delta < \delta_1$ and $|\varphi(s) - \varphi(s')| < \delta_2$ whenever $|s - s'| < \delta$ (by the definition of uniform continuity). So if $P_2 = \{c = s_0 < s_1 < \cdots < s_n = d\}$ is a partition of $[c, d]$ with $\|P_3\| < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $P_4 = \{a = t_0 \leq t_1 \leq \cdots \leq t_n = b\}$ is a partition of $[a, b]$ with $\|P_4\| < \delta_2$. If $s_{k-1} \leq \sigma_k \leq s_k$ and $\tau_k = \varphi(\sigma_k)$ then both above inequalities hold and

$$\left|\int_\gamma f - \int_{\gamma \circ \sigma} f\right| = \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] + \sum_{k=1}^n f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \varphi(s_k) - \gamma \circ (\varphi(s_{k-1})) - \int_{\gamma \circ \sigma}] < \varepsilon \text{ and the result follows.}$$
Proposition IV.1.13. If \( \gamma : [a, b] \to \mathbb{C} \) is a rectifiable path and \( \sigma : [c, d] \to [a, b] \) is a continuous non-decreasing function with \( \sigma(c) = a \) and \( \sigma(d) = b \), then for any \( f \) continuous on \( \{\gamma\} = \{\gamma \circ \sigma\} \) we have \( \int_{\gamma} f = \int_{\gamma \circ \sigma} f \).

Proof (continued). Since \( \varphi \) is continuous on \([c, d]\) and \([c, d]\) is compact, then there is a \( \delta > 0 \) such that \( \delta < \delta_1 \) and \( |\varphi(s) - \varphi(s')| < \delta_2 \) whenever \( |s - s'| < \delta \) (by the definition of uniform continuity). So if \( P_2 = \{c = s_0 < s_1 < \cdots < s_n = d\} \) is a partition of \([c, d]\) with \( \|P_3\| < \delta < \delta_1 \) and \( t_k = \varphi(s_k) \), then \( P_4 = \{a = t_0 \leq t_1 \leq \cdots \leq t_n = b\} \) is a partition of \([a, b]\) with \( \|P_4\| < \delta_2 \). If \( s_{k-1} \leq \sigma_k \leq s_k \) and \( \tau_k = \varphi(\sigma_k) \) then both above inequalities hold and

\[
\left| \int_{\gamma} f - \int_{\gamma \circ \sigma} f \right| = \left| \int_{\gamma} f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right|
+ \sum_{k=1}^{n} f(\gamma \circ \varphi(\sigma_k))[\gamma \circ \varphi(s_k) - \gamma \circ (\varphi(s_{k-1})] - \int_{\gamma \circ \sigma} f < \varepsilon \text{ and the result follows.}
\]
Lemma IV.1.19. If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon$.

Proof. Case I. Suppose $G$ is an open disk.
Lemma IV.1.19

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

**Proof.** Case I. Suppose $G$ is an open disk. Since $\{\gamma\}$ is a compact set, by Theorem II.5.17, $d = \text{dist}(\{\gamma\}, \partial(G)) > 0$ where $\partial(G)$ is the boundary of $G$. 
**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and
\[
\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.
\]

**Proof. Case I.** Suppose $G$ is an open disk. Since $\{\gamma\}$ is a compact set, by Theorem II.5.17, $d = \text{dist}(\{\gamma\}, \partial(G)) > 0$ where $\partial(G)$ is the boundary of $G$. So if $G = B(c; r)$ then $\{\gamma\} \subset B(c; \rho)$ where $\rho = r - \frac{1}{2} d$:  

Lemma IV.1.19

Lemma IV.1.19. If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon$.

Proof. Case I. Suppose $G$ is an open disk. Since $\{\gamma\}$ is a compact set, by Theorem II.5.17, $d = \text{dist}(\{\gamma\}, \partial(G)) > 0$ where $\partial(G)$ is the boundary of $G$. So if $G = B(c; r)$ then $\{\gamma\} \subset B(c; \rho)$ where $\rho = r - \frac{1}{2}d$: 

$G = B(c; r)$

$\{\gamma\}$

$B(c; \rho)$
Lemma IV.1.19. If \( G \) is an open set in \( \mathbb{C} \), \( \gamma : [a, b] \to G \) is a rectifiable path, and \( f : G \to \mathbb{C} \) is continuous then for every \( \varepsilon > 0 \) there is a polygonal path \( \Gamma \) in \( G \) such that \( \Gamma(a) = \gamma(a) \), \( \Gamma(b) = \gamma(b) \), and

\[
\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.
\]

Proof. Case I. Suppose \( G \) is an open disk. Since \( \{\gamma\} \) is a compact set, by Theorem II.5.17, \( d = \text{dist}(\{\gamma\}, \partial(G)) > 0 \) where \( \partial(G) \) is the boundary of \( G \). So if \( G = B(c; r) \) then \( \{\gamma\} \subset B(c; \rho) \) where \( \rho = r - \frac{1}{2}d \):
Proof. Case I (continued 1). Now $f$ is uniformly continuous on $\overline{B}(c; \rho) \subset G$ since $\overline{B}(c; \rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a, b]$ and so $\gamma$ is also uniformly continuous.
Lemma IV.1.19, Case I (continued)

**Proof. Case I (continued 1).** Now $f$ is uniformly continuous on $\overline{B}(c; \rho) \subset G$ since $\overline{B}(c; \rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a, b]$ and so $\gamma$ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for $s, t$ such that $t_{k-1} \leq s \leq t_k$ and $t_{k-1} \leq t \leq t_k$, and (2) for $\tau_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon \quad (1.20)$$

(by the definition of $\int_\gamma f$).
Lemma IV.1.19, Case I (continued)

**Proof. Case I (continued 1).** Now $f$ is uniformly continuous on $\overline{B}(c; \rho) \subset G$ since $\overline{B}(c; \rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a, b]$ and so $\gamma$ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for $s, t$ such that $t_{k-1} \leq s \leq t_k$ and $t_{k-1} \leq t \leq t_k$, and (2) for $t_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_{\gamma} f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right| < \varepsilon \quad (1.20)$$

(by the definition of $\int_{\gamma} f$). We now use this partition of $[a, b]$ to define the desired polygon. Define $\Gamma : [a, b] \to \mathbb{C}$ as

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}}[(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)] \quad \text{for} \ t \in [t_{k-1}, t_k]$$
Lemma IV.1.19, Case I (continued)

Proof. Case I (continued 1). Now $f$ is uniformly continuous on $\overline{B}(c; \rho) \subset G$ since $\overline{B}(c; \rho)$ is compact. So WLOG, $f$ is uniformly continuous on $G$. Choose $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$. $\gamma$ is defined on $[a, b]$ and so $\gamma$ is also uniformly continuous. So there is a partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$ such that the norm of this partition is sufficiently small so that (1) $|\gamma(s) - \gamma(t)| < \delta/2$ for $s, t$ such that $t_{k-1} \leq s \leq t_k$ and $t_{k-1} \leq t \leq t_k$, and (2) for $\tau_k \in [t_{k-1}, t_k]$ we have

$$\left| \int_{\gamma} f - \sum_{k=1}^{n} f(\gamma(\tau_k))[(\gamma(t_k) - \gamma(t_{k-1}))] \right| < \varepsilon \quad (1.20)$$

(by the definition of $\int_{\gamma} f$). We now use this partition of $[a, b]$ to define the desired polygon. Define $\Gamma : [a, b] \rightarrow \mathbb{C}$ as

$$\Gamma(t) = \frac{1}{t_k - t_{k-1}}[(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)] \text{ for } t \in [t_{k-1}, t_k]$$
Lemma IV.1.19, Case I (continued 2)

Proof. Case I. (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))]$. Then $\Gamma$ is a polygonal path and a subset of $G$ (since $G$ is convex; it’s a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \leq s \leq t \leq t_k$, then

$$|\Gamma(t) - \gamma(\tau_k)| = |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)|$$

$$\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21)$$

for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$:}
Lemma IV.1.19, Case I (continued 2)

**Proof. Case I.** (so \( \Gamma(t_{k-1}) = \gamma(t_{k-1}) \), \( \Gamma(t_k) = \gamma(t_k) \), and hence \( \Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))] \). Then \( \Gamma \) is a polygonal path and a subset of \( G \) (since \( G \) is convex; it’s a disk). Since \( |\gamma(s) - \gamma(t)| < \delta/2 \) for \( t_{k-1} \leq s \leq t \leq t_k \), then

\[
|\Gamma(t) - \gamma(\tau_k)| = |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)| \\
\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21)
\]

for \( t \in [t_{k-1}, t_k] \) (\( \Gamma(t) \) is at least as close to \( \gamma(t_k) \) as \( \gamma(t_{k-1}) \) is, and so the distance \( |\Gamma(t) - \gamma(t_k)| \) is less than \( \delta/2 \):
Lemma IV.1.19, Case I (continued 2)

**Proof. Case I.** (so $\Gamma(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma(t_k) = \gamma(t_k)$, and hence $\Gamma([t_{k-1}, t_k]) = [\gamma(t_{k-1}, \gamma(t_k))]$. Then $\Gamma$ is a polygonal path and a subset of $G$ (since $G$ is convex; it’s a disk). Since $|\gamma(s) - \gamma(t)| < \delta/2$ for $t_{k-1} \leq s \leq t \leq t_k$, then

$$|\Gamma(t) - \gamma(\tau_k)| = |\Gamma(t) - \gamma(t_k) + \gamma(t_k) - \gamma(\tau_k)|$$

$$\leq |\Gamma(t) - \gamma(t_k)| + |\gamma(t_k) - \gamma(\tau_k)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad (1.21)$$

for $t \in [t_{k-1}, t_k]$ ($\Gamma(t)$ is at least as close to $\gamma(t_k)$ as $\gamma(t_{k-1})$ is, and so the distance $|\Gamma(t) - \gamma(t_k)|$ is less than $\delta/2$:

$$\gamma(t_{k-1}) \quad \Gamma(t) \quad \gamma(t_k)$$
Lemma IV.1.19, Case I (continued 3)

**Proof. Case I.** Since \( \int_{\Gamma} f = \int_{a}^{b} f(\Gamma(t)) \Gamma'(t) \, dt \) (computed piecewise), then

\[
\int_{\Gamma} f = \sum_{k=1}^{n} \left( \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right) \int_{t_{k-1}}^{t_k} f(\Gamma(t)) \, dt.
\]

Next,

\[
\left| \int_{\gamma} f - \int_{\Gamma} f \right| = \left| \int_{\gamma} f - \sum_{k=1}^{n} f(\gamma(t_k)) [\gamma(t_k) - \gamma(t_{k-1})] + \sum_{k=1}^{n} f(\gamma(t_k)) [\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \right| < \varepsilon + \sum_{k=1}^{m} f(\gamma(t_k)) [\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \quad \text{by (1.20)}
\]
Lemma IV.1.19, Case I (continued 3)

**Proof. Case I.** Since $\int_{\Gamma} f = \int_{a}^{b} f(\Gamma(t))\Gamma'(t)\,dt$ (computed piecewise), then

$$
\int_{\Gamma} f = \sum_{k=1}^{n} \left( \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right) \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,dt.
$$

Next,

$$
\left| \int_{\gamma} f - \int_{\Gamma} f \right| = \left| \int_{\gamma} f - \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] \right|
$$

$$
+ \sum_{k=1}^{n} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f
$$

$$
< \varepsilon + \left| \sum_{k=1}^{m} f(\gamma(\tau_k))[\gamma(t_k) - \gamma(t_{k-1})] - \int_{\Gamma} f \right| \text{ by (1.20)}
$$
Lemma IV.1.19, Case I (continued 4)

Proof. Case I.

\[
\left| \int_\gamma f - \int_\Gamma f \right| = \varepsilon + \left| \sum_{k=1}^{n} \left( f(\gamma(\tau_k))\left[ \gamma(t_k) - \gamma(t_{k-1}) \right] \right) \right|
\]

\[
- \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \left| \int_{t_{k-1}}^{t_k} f(\Gamma(t)) \, dt \right|
\]

\[
= \varepsilon + \left| \sum_{k=1}^{n} \left( \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \right) \int_{t_{k-1}}^{t_k} \left( f(\gamma(\tau_k)) - f(\Gamma(t)) \right) \, dt \right|
\]

\[
\leq \varepsilon + \sum_{k=1}^{n} \left( \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))| \, dt \right).
\]
Lemma IV.1.19, Case I (continued 5)

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

**Proof. Case I.** To recap: $G$ is an open disk and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| \leq \varepsilon + \sum_{k=1}^{n} \left( \frac{\left| \gamma(t_k) - \gamma(t_{k-1}) \right|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau_k)) - f(\Gamma(t))| \, dt \right).$$

By (1.21), $|\Gamma(t) - \gamma(\tau_k)| < \delta$ and by uniform continuity mentioned above, $|f(\gamma(\tau_k)) - f(\Gamma(t))| < \varepsilon$, so

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon = \varepsilon \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \leq \varepsilon (1 - V(\gamma)).$$

Since $\varepsilon$ is arbitrary, Case I follows.
Lemma IV.1.19, Case I (continued 5)

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

**Proof. Case I.** To recap: $G$ is an open disk and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| \leq \varepsilon + \sum_{k=1}^{n} \left( \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\gamma(\tau)) - f(\Gamma(t))| \, dt \right).$$

By (1.21), $|\Gamma(t) - \gamma(\tau_k)| < \delta$ and by uniform continuity mentioned above, $|f(\gamma(\tau_k)) - f(\Gamma(t))| < \varepsilon$, so

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Since $\varepsilon$ is arbitrary, Case I follows.
Lemma IV.1.19, Case II

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and $\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon$.

**Proof.** Case II. $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. 


Lemma IV.1.19, Case II

Lemma IV.1.19. If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and 
\[ \left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon. \]

Proof. Case II. $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of $\gamma$ on $[a, b]$). If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the “$k$th part” of $\gamma$ contained in $B(\gamma(t_{k-1}); r)$ and can use Case I.
Lemma IV.1.19, Case II

**Lemma IV.1.19.** If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \rightarrow G$ is a rectifiable path, and $f : G \rightarrow \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

\[ \left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon. \]

**Proof.** Case II. $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of $\gamma$ on $[a, b]$). If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the “$k$th part” of $\gamma$ contained in $B(\gamma(t_{k-1}); r)$ and can use Case I. If $\gamma_k : [t_{k-1}, t_k] \rightarrow G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \leq k \leq n$ (the “parts” of $\gamma$). By Case I there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \rightarrow B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $\left| \int_{\gamma_k} f - \int_{\Gamma_k} f \right| < \varepsilon/n$. Defining $\Gamma$ as the union of the $\Gamma_k$ yields the desired polygonal path.
Lemma IV.1.19, Case II

Lemma IV.1.19. If $G$ is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path $\Gamma$ in $G$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon.$$

Proof. Case II. $G$ is an arbitrary set. As in Case I, since $\{\gamma\}$ is compact there is a number $r$ such that $0 < r < \text{dist}(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$ (by the uniform continuity of $\gamma$ on $[a, b]$). If $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ with $\|P\| < \delta$ then $|\gamma(t) - \gamma(t_{k-1})| < r$ for $t \in [t_{k-1}, t_k]$. So we now have the “$k$th part” of $\gamma$ contained in $B(\gamma(t_{k-1}); r)$ and can use Case I. If $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$ then $\{\gamma_k\} \subset B(\gamma(t_{k-1}); r)$ for $1 \leq k \leq n$ (the “parts” of $\gamma$). By Case I there is a polygonal path $\Gamma_k : [t_{k-1}, t_k] \to B(\gamma(t_{k-1}); r)$ such that $\Gamma_k(t_{k-1}) = \gamma(t_{k-1})$, $\Gamma_k(t_k) = \gamma(t_k)$, and $|\int_{\gamma_k} f - \int_{\Gamma_k} f| < \varepsilon/n$. Defining $\Gamma$ as the union of the $\Gamma_k$ yields the desired polygonal path.
Theorem IV.1.18. Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

Proof. Case I. Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth.
Theorem IV.1.18

**Theorem IV.1.18.** Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a *primitive* $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_\gamma f = F(\beta) - F(\alpha)$.

**Proof. Case I.** Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth. Then

\[
\int_\gamma f = \int_a^b f(\gamma(t)) \gamma'(t) \, dt \text{ (piecewise)}
\]

\[
= \int_a^b F'(\gamma(t)) \gamma'(t) \, dt = \int_a^b (F \circ \gamma)'(t) \, dt
\]

\[
= \int_a^b \text{Re}((F \circ \gamma)') \, dt + i \int_a^b \text{Im}((F \circ \gamma)') \, dt
\]

\[
= \text{Re}((F \circ \gamma))|_a^b + i \text{Im}((F \circ \gamma))|_a^b \text{ by the F.T.C.}
\]

\[
= F(\gamma(b)) - F(\gamma(a)).
\]
Theorem IV.1.18

**Theorem IV.1.18.** Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

**Proof. Case I.** Suppose $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth. Then

$$
\int_{\gamma} f = \int_{a}^{b} f(\gamma(t))\gamma'(t) \, dt \quad \text{(piecewise)}
$$

$$
= \int_{a}^{b} F'(\gamma(t))\gamma'(t) \, dt = \int_{a}^{b} (F \circ \gamma)'(t) \, dt
$$

$$
= \int_{a}^{b} \Re\{(F \circ \gamma)'\} \, dt + i \int_{a}^{b} \Im\{(F \circ \gamma)'\} \, dt
$$

$$
= \Re\{(F \circ \gamma)\}|_{a}^{b} + i\Im\{(F \circ \gamma)\}|_{a}^{b} \text{ by the F.T.C.}
$$

$$
= F(\gamma(b)) - F(\gamma(a)).
$$
Theorem IV.1.18. Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_{\gamma} f = F(\beta) - F(\alpha)$.

Proof. Case II. Suppose $\gamma$ is rectifiable. For $\varepsilon > 0$, Lemma IV.1.19 implies there is a polygonal path $\Gamma$ from $\alpha$ to $\beta$ such that $\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \varepsilon$. 
Theorem IV.1.18. Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_\gamma f = F(\beta) - F(\alpha)$.

**Proof.** Case II. Suppose $\gamma$ is rectifiable. For $\varepsilon > 0$, Lemma IV.1.19 implies there is a polygonal path $\Gamma$ from $\alpha$ to $\beta$ such that $\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$. But $\Gamma$ is piecewise smooth, so by Case I, $\int_\Gamma f = F(\beta) - F(\alpha)$. Therefore $\left| \int_\gamma f - [F(\beta) - F(\alpha)] \right| < \varepsilon$, and the result follows.
Theorem IV.1.18. Let $G$ be open in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (i.e., $F' = f$), then $\int_\gamma f = F(\beta) - F(\alpha)$.

**Proof.** Case II. Suppose $\gamma$ is rectifiable. For $\varepsilon > 0$, Lemma IV.1.19 implies there is a polygonal path $\Gamma$ from $\alpha$ to $\beta$ such that $\left| \int_\gamma f - \int_\Gamma f \right| < \varepsilon$. But $\Gamma$ is piecewise smooth, so by Case I, $\int_\Gamma f = F(\beta) - F(\alpha)$. Therefore $\left| \int_\gamma f - [F(\beta) - F(\alpha)] \right| < \varepsilon$, and the result follows. $\square$