Proposition IV.2.1

**Proposition IV.2.1.** Let \( \varphi : [a, b] \times [c, d] \rightarrow \mathbb{C} \) be a continuous function and define \( g : [c, d] \rightarrow \mathbb{C} \) by \( g(t) = \int_a^b \varphi(s, t) \, ds \). Then \( g \) is continuous.

Moreover, if \( \frac{\partial \varphi}{\partial t} \) exists and is a continuous function on \([a, b] \times [c, d]\) then \( g \) is continuously differentiable and
\[
g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds.
\]

**Proof.** The proof that \( g \) is continuous is left as Exercise IV.2.1.

Now suppose \( \frac{\partial \varphi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\). Since \([a, b] \times [c, d]\) is a compact subset of \( \mathbb{R}^2 \) then by Theorem II.5.15, \( \frac{\partial \varphi}{\partial t} \) is uniformly continuous on \([a, b] \times [c, d]\). Now denote \( \frac{\partial \varphi}{\partial t} = \varphi_2 \). Fix a point \( t_0 \in [c, d] \) and let \( \varepsilon > 0 \). So there is \( \delta > 0 \) such that
\[
|\varphi_2(s', t') - \varphi_2(s, t)| < \varepsilon \text{ whenever } (s - s')^2 + (t - t')^2 < \delta^2.
\]
Proposition IV.2.1 (continued 3)

**Proposition IV.2.1.** Let \( \varphi : [a, b] \times [c, d] \to \mathbb{C} \) be a continuous function and define \( g : [c, d] \to \mathbb{C} \) by \( g(t) = \int_a^b \varphi(s, t) \, ds \). Then \( g \) is continuous. Moreover, if \( \frac{\partial \varphi}{\partial t} \) exists and is a continuous function on \([a, b] \times [c, d]\) then \( g \) is continuously differentiable and

\[
g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds.
\]

**Proof (continued).** Since \( t_0 \) is an arbitrary element of \([c, d]\) then we have \( g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds \) on \([a, b] \times [c, d]\), as claimed. Since \( \frac{\partial \varphi}{\partial t} \) is hypothesized to be continuous then \( g' \) is continuous by Exercise IV.2.1 (with \( g \) and \( \varphi \) of the exercise replaced with \( g' \) and \( \partial \varphi/\partial t \) here), as claimed.

Lemma IV.2.A

**Lemma IV.2.A.** If \( |z| < 1 \) then

\[
\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi.
\]

**Proof.** Let \( \varphi(s, t) = \frac{e^{is}}{e^{is} - tz} \) for \( 0 \leq t \leq 1 \) and \( 0 \leq s \leq 2\pi \). Since \( |z| < 1 \), \( \varphi \) is continuously differentiable. So by Proposition IV.2.1, \( g(t) = \int_0^{2\pi} \varphi(s, t) \, ds \) is continuously differentiable. Also,

\[
g(0) = \int_0^{2\pi} \varphi(s, 0) \, ds = -\int_0^{2\pi} \frac{e^{is}}{e^{is} - 0} \, dz = \int_0^{2\pi} 1 \, dz = 2\pi.
\]

Next, \( g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds \) by Proposition IV.2.1. Notice for \( \Phi(s) = \frac{zi}{e^{is} - tz} \) (with \( t \) fixed) we have \( \Phi'(s) = \frac{ze^{is}}{(e^{is} - tz)^2} \) and so \( \Phi(s) \) is a primitive for \( \frac{ze^{is}}{(e^{is} - tz)^2} \), and so

Lemma IV.2.A (continued)

**Lemma IV.2.A.** If \( |z| < 1 \) then

\[
\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi.
\]

**Proof (continued).**

\[
g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds = \Phi(2\pi) - \Phi(0) = \frac{zi}{e^{2\pi i} - tz} - \frac{z}{e^0 - tz} = 0.
\]

Therefore \( g \) is constant and \( g(1) = g(0) = 2\pi \). That is,

\[
g(1) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, dz = 2\pi.
\]

Theorem IV.2.6

**Proposition IV.2.6.** Let \( f : G \to \mathbb{C} \) be analytic and suppose \( \overline{B(a; r)} \subseteq G \) (\( r > 0 \)). If \( \gamma(t) = a + re^{it} \), and \( 0 \leq t \leq 2\pi \). Then

\[
f(z) = \left. \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \right|_{z = a}
\]

for \( |z - a| < r \).

**Proof.** Without loss of generality, we assume \( a = 0 \) and \( r = 1 \) (otherwise, we consider \( g(z) = f(a + rz) \) and \( G_1 = \{ \frac{1}{r}(z - a) \mid z \in G \} \)). That is, \( \overline{B(0, 1)} \subseteq G \). Fix \( z \) where \( |z| < 1 \). We then need to show that

\[
f(z) = \left. \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \right|_{z = a} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z} \, ds.
\]

This is equivalent to

\[
0 = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z} \, ds - 2\pi f(z) = \int_0^{2\pi} \left( \frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) \, ds.
\]
Theorem IV.2.6 (continued)

Proof (continued). Let \( \varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z) \) for \( 0 \leq t \leq 1 \) and \( 0 \leq s \leq 2\pi \). Since

\[ |z + t(e^{is} - z)| = |z(1 - t) + te^{is}| \leq |z(1 - t)| + t \leq |1 - t| + t = 1 - t + t = 1, \]

then \( \varphi \) is well defined (\( f \) takes on values in \( B(0; 1) \subset G \)) and is continuously differentiable. Let \( g(t) = \int_0^{2\pi} \varphi(s, t) \, ds \). Then by Proposition IV.2.1, \( g \) is continuously differentiable. Notice that

\[
g(0) = \int_0^{2\pi} \varphi(s, 0) \, ds = \int_0^{2\pi} \left( \frac{f(z)e^{is}}{e^{is} - z} - f(z) \right) \, ds
= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds - 2\pi f(z)
= 0 \text{ by Lemma IV.2.6}
\]

We now show \( g \) is constant. By Proposition IV.2.1,

\[ g'(t) = \int_0^{2\pi} \varphi_2(s, t) \, ds \text{ where } \varphi_2(s, t) = e^{is}f'(z + t(e^{is} - z)) = \partial \varphi/\partial t. \]

This is (*) and the result follows.

Lemma IV.2.7

Lemma IV.2.7. Let \( \gamma \) be a rectifiable curve in \( \mathbb{C} \) and suppose that \( F_n \) and \( F \) are continuous on \( \{ \gamma \} \). If \( F \) is the uniform limit of \( F_n \) on \( \{ \gamma \} \) then

\[
\int_{\gamma} F = \lim \left( \int_{\gamma} F_n \right).
\]

Proof. Let \( \varepsilon > 0 \); then there is \( N \in \mathbb{N} \) such that

\[
|F_n(w) - F(w)| < \varepsilon/V(\gamma) \text{ for all } w \in \{ \gamma \} \text{ and } n \geq N.
\]

Then

\[
\left| \int_{\gamma} F - \int_{\gamma} F_n \right| 
= \left| \int_{\gamma} (F - F_n) \right|
\leq \int_{\gamma} |F(w) - F_n(w)| \, |dw| \text{ by Proposition IV.1.17}
< \frac{\varepsilon}{V(\gamma)} V(\gamma) = \varepsilon
\]

for all \( n \geq N \). So \( \int_{\gamma} F = \lim(\int_{\gamma} F_n) \).

Theorem IV.2.8

Theorem IV.2.8. Let \( f \) be analytic in \( B(a; R) \). Then

\[ f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n \text{ for } |z - a| < R \text{ where } a_n = f^{(n)}(a)/n! \text{ and this series has radius of convergence } \geq R. \]

Proof. Let \( 0 < r < R \) and then \( \overline{B}(a; r) \subset B(a; R) \). If \( \gamma(t) = a + re^{it}, \)
\( t \in [0, 2\pi], \) then by Proposition IV.2.6, \( f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \) for

\[ |z - a| < r. \]

For \( |z - a| < r \) and \( w \in \{ \gamma \}, \)

\[ |f(w)| \left| \frac{z - a}{w - a} \right|^n \leq M \left( \frac{|z - a|}{r} \right)^n \]

where \( M = \max \{|f(w)| \mid |w - a| = r \}. \)

Since \( |z - a|/r < 1 \), the Weierstrass M-Test (with \( M_n = (|z - a|/r)^n \)) implies that \( \sum_{n=1}^{\infty} f(w)(z - a)^n/(w - a)^{n+1} \) converges uniformly for \( w \in \{ \gamma \}. \)
Theorem 4.2.8

Proof (continued). From the computation before Lemma 4.2.7 we have

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw \text{ by Proposition 4.2.6} \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left( \frac{z-a}{w-a} \right)^n \right) \, dw \]
\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \right) (z-a)^n \text{ by Lemma 4.2.7.} \]

Next set \( a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \) and we have

\[ f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \] where the series converges if \( |z-a| < r \). By Proposition 3.2.5, \( a_n = \frac{f^{(n)}(a)}{n!} \). So each \( a_n \) is (1) independent of \( z \),

(2) independent of \( \{\gamma\} \), and (3) independent of \( r \). Since \( r \) was chosen arbitrarily and \( < R \), then the series representation holds for all \( z \) such that \( |z-a| < R \) and the radius of convergence of the series is at least \( R \). \( \square \)

Theorem 4.2.14

**Theorem 4.2.14. Cauchy’s Estimate.** Let \( f \) be analytic in \( B(a; R) \) and suppose \( |f(z)| \leq M \) for all \( z \in B(a; R) \). Then

\[ |f^{(n)}(a)| \leq \frac{n! M}{R^n}. \]

**Proof.** By Corollary 4.2.13, for \( r < R \) we have

\[ |f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \right| \text{ where } \gamma(t) = a + re^{it}, t \in [0, 2\pi] \]
\[ \leq \frac{n!}{2\pi} \int_{\gamma} \left| \frac{f(w)}{(w-a)^{n+1}} \right| \, dw \text{ by Proposition 4.1.17(b)} \]
\[ \leq \frac{n! M}{2\pi r^{n+1}} (2\pi r) \text{ by Proposition 4.1.17(b)} \]
\[ = \frac{n! M}{r^n}. \]

Now let \( r \to R^- \) and the result follows. \( \square \)

Theorem 4.2.15

**Proposition 4.2.15.** Let \( f \) be analytic in \( B(a; R) \) and suppose \( \gamma \) is a closed rectifiable curve in \( B(a; R) \). Then \( f \) has a primitive in \( B(a; R) \) and so \( \int_{\gamma} f = 0 \).

**Proof.** We know by Theorem 4.2.8, that an analytic function has a power series representation: \( f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \) for \( z \in B(a; R) \). Define

\[ F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^{n+1} = (z-a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^n. \]

Then, by definition, the radius of convergence of \( F \) is

\[ \lim_{n \to \infty} \frac{1}{\left( \frac{a_n}{n+1} \right)^{1/n}} = \lim_{n \to \infty} \frac{1}{\left( \frac{a_n}{n+1} \right)^{1/n}} = \lim_{n \to \infty} \frac{1}{a_n^{1/n}} \]

and so the radius of convergence of \( F \) is the same as the radius of convergence of \( f \). So \( F \) is defined on \( B(a; R) \). Also, by Proposition 3.2.5, \( F'(z) = f(z) \). So \( F \) is a primitive of \( f \) and by Corollary 4.1.22,

\[ \int_{\gamma} f = 0. \]