Chapter IV. Complex Integration
IV.2. Power Series Representation of Analytic Functions—Proofs
Proposition IV.2.1

Proposition IV.2.1. Let \( \varphi : [a, b] \times [c, d] \to \mathbb{C} \) be a continuous function and define \( g : [c, d] \to \mathbb{C} \) by \( g(t) = \int_a^b \varphi(s, t) \, ds \). Then \( g \) is continuous.

Moreover, if \( \frac{\partial \varphi}{\partial t} \) exists and is a continuous function on \( [a, b] \times [c, d] \) then \( g \) is continuously differentiable and

\[
g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds.
\]

Proof. The proof that \( g \) is continuous is left as Exercise IV.2.1.
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**Proof.** The proof that \( g \) is continuous is left as Exercise IV.2.1.

Now suppose \( \frac{\partial \varphi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d] \). Since \([a, b] \times [c, d] \) is a compact subset of \( \mathbb{R}^2 \) then by Theorem II.5.15, \( \frac{\partial \varphi}{\partial t} \) is uniformly continuous on \([a, b] \times [c, d] \). Now denote \( \frac{\partial \varphi}{\partial t} = \varphi_2 \). Fix a point \( t_0 \) is \([c, d] \) and let \( \varepsilon > 0 \). So there is \( \delta > 0 \) such that

\[
|\varphi_2(s', t') - \varphi_2(s, t)| < \varepsilon \text{ whenever } (s - s')^2 + (t - t')^2 < \delta^2.
\]
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Proof. The proof that \( g \) is continuous is left as Exercise IV.2.1.

Now suppose \( \frac{\partial \varphi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\). Since \([a, b] \times [c, d]\) is a compact subset of \( \mathbb{R}^2 \) then by Theorem II.5.15, \( \frac{\partial \varphi}{\partial t} \) is uniformly continuous on \([a, b] \times [c, d]\). Now denote \( \frac{\partial \varphi}{\partial t} = \varphi_2 \). Fix a point \( t_0 \) is \([c, d]\) and let \( \varepsilon > 0 \). So there is \( \delta > 0 \) such that

\[
|\varphi_2(s', t') - \varphi_2(s, t)| < \varepsilon \quad \text{whenever} \quad (s - s')^2 + (t - t')^2 < \delta^2.
\]
Proposition IV.2.1 (continued 1)

**Proof (continued).** In particular, $|\varphi_2(s, t) - \varphi_2(s, t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$ and $s \in [a, b]$. So for $|t - t_0| < \delta$ and $x \in [a, b]$ we have

$$\left| \int_{t_0}^{t} (\varphi_2(s, \tau) - \varphi_2(s, t_0)) d\tau \right| \leq \varepsilon|t - t_0|.$$ 

But for a fixed $s \in [a, b]$, $\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$ is a primitive of $\varphi_2(s, t) - \varphi_2(s, t_0)$, so by the Fundamental Theorem of Calculus we have

$$\left| \int_{t_0}^{t} (\varphi_2(s, \tau) - \varphi_2(s, t_0)) d\tau \right|$$

$$= \left| (\varphi(s, t) - t\varphi_2(s, t_0)) - (\varphi(s, t_0) - t_0\varphi_2(s, t_0)) \right|$$

$$= \left| \varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0) \right| \leq \varepsilon|t - t_0|$$

and this holds for any $s \in [a, b]$ when $|t - t_0| < \delta$. 
Proposition IV.2.1 (continued 1)

Proof (continued). In particular, $|\varphi_2(s, t) - \varphi_2(s, t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$ and $s \in [a, b]$. So for $|t - t_0| < \delta$ and $x \in [a, b]$ we have

$$\left| \int_{t_0}^{t} (\varphi_2(s, \tau) - \varphi_2(s, t_0)) \, d\tau \right| \leq \varepsilon |t - t_0|.$$ 

But for a fixed $s \in [a, b]$, $\Phi(t) = \varphi(s, t) - t\varphi_2(s, t_0)$ is a primitive of $\varphi_2(s, t) - \varphi_2(s, t_0)$, so by the Fundamental Theorem of Calculus we have

$$\left| \int_{t_0}^{t} (\varphi_2(s, \tau) - \varphi_2(s, t_0)) \, d\tau \right|$$

$$= \left| (\varphi(s, t) - t\varphi_2(s, t_0)) - (\varphi(s, t_0) - t_0\varphi_2(s, t_0)) \right|$$

$$= \left| \varphi(s, t) - \varphi(s, t_0) - (t - t_0)\varphi_2(s, t_0) \right| \leq \varepsilon |t - t_0|$$

and this holds for any $s \in [a, b]$ when $|t - t_0| < \delta$. 
Proposition IV.2.1 (continued 2)

Proof (continued). Therefore for $s \in [a, b]$ and $|t - t_0| < \delta$ we have

\[
\left| \frac{\varphi(s, t) - \varphi(s, t_0)}{t - t_0} - \varphi_2(s, t_0) \right| \leq \varepsilon \text{ and }
\]

\[
\left| \int_a^b \frac{\varphi(s, t) - \varphi(s, t_0)}{t - t_0} \, ds - \int_a^b \varphi_2(s, t_0) \, ds \right| \leq \varepsilon (b - a) \text{ or }
\]

\[
\left| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \varphi_2(s, t_0) \, ds \right| \leq \varepsilon (b - a)
\]

since $g(t) = \int_a^b \varphi(s, t) \, ds$ by definition. Therefore for $s \in [a, b]$ we have

\[
g'(t_0) = \int_a^b \varphi_2(s, t_0) \, ds = \int_a^b \frac{\partial \varphi}{\partial t}(s, t_0) \, ds.
\]
Proposition IV.2.1 (continued 3)

Proposition IV.2.1. Let $\varphi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function and define $g : [c, d] \to \mathbb{C}$ by $g(t) = \int_{a}^{b} \varphi(s, t) \, ds$. Then $g$ is continuous.

Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$ then $g$ is continuously differentiable and

$$g'(t) = \int_{a}^{b} \frac{\partial \varphi}{\partial t}(s, t) \, ds.$$

Proof (continued). Since $t_0$ is an arbitrary element of $[c, d]$ then we have

$$g'(t) = \int_{a}^{b} \frac{\partial \varphi}{\partial t}(s, t) \, ds$$

on $[a, b] \times [c, d]$, as claimed. Since $\frac{\partial \varphi}{\partial t}$ is hypothesized to be continuous then $g'$ is continuous by Exercise IV.2.1 (with $g$ and $\varphi$ of the exercise replaced with $g'$ and $\frac{\partial \varphi}{\partial t}$ here), as claimed.
Lemma IV.2.A

Lemma IV.2.A. If $|z| < 1$ then \[ \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi. \]

Proof. Let $\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z| < 1$, $\varphi$ is continuously differentiable. So by Proposition IV.2.1, $g(t) = \int_0^{2\pi} \varphi(s, t) \, ds$ is continuously differentiable.
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Proof. Let $\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z| < 1$, $\varphi$ is continuously differentiable. So by Proposition IV.2.1, $g(t) = \int_0^{2\pi} \varphi(s, t) \, ds$ is continuously differentiable. Also,

$$g(0) = \int_0^{2\pi} \varphi(s, 0) \, ds = \int_0^{2\pi} \frac{e^{is}}{e^{is} - 0z} \, dz = \int_0^{2\pi} 1 \, dz = 2\pi.$$

Next, $g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds$ by Proposition IV.2.1.
Lemma IV.2.A. If $|z| < 1$ then \[ \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi. \]

**Proof.** Let \( \varphi(s, t) = \frac{e^{is}}{e^{is} - tz} \) for \( 0 \leq t \leq 1 \) and \( 0 \leq s \leq 2\pi \). Since \( |z| < 1 \), \( \varphi \) is continuously differentiable. So by Proposition IV.2.1, \( g(t) = \int_0^{2\pi} \varphi(s, t) \, ds \) is continuously differentiable. Also,

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g(0) = \int_0^{2\pi} \varphi(s, 0) \, ds = \int_0^{2\pi} \frac{e^{is}}{e^{is} - 0z} \, dz = \int_0^{2\pi} 1 \, dz = 2\pi.
\]

Next, \( g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds \) by Proposition IV.2.1. Notice for \( \Phi(s) = \frac{zi}{e^{is} - tz} \) (with \( t \) fixed) we have \( \Phi'(s) = \frac{ze^{is}}{(e^{is} - tz)^2} \) and so \( \Phi(s) \) is a primitive for \( \frac{ze^{is}}{(e^{is} - tz)^2} \), and so
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Proof. Let $\varphi(s, t) = \frac{e^{is}}{e^{is} - tz}$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z| < 1$, $\varphi$ is continuously differentiable. So by Proposition IV.2.1, $g(t) = \int_0^{2\pi} \varphi(s, t) \, ds$ is continuously differentiable. Also,

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Lemma IV.2.A. If $|z| < 1$ then \[ \int_{0}^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi. \]

Proof (continued).

\[
g'(t) = \int_{0}^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds = \Phi(2\pi) - \Phi(0) = \frac{zi}{e^{2\pi i} - tz} - \frac{z}{e^{0} - tz} = 0.
\]

Therefore $g$ is constant and $g(1) = g(0) = 2\pi$. 
Lemma IV.2.A. If $|z| < 1$ then 
\[ \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi. \]

Proof (continued).

\[ g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2} \, ds = \Phi(2\pi) - \Phi(0) = \frac{zi}{e^{2\pi i} - tz} - \frac{z}{e^0 - tz} = 0. \]

Therefore $g$ is constant and $g(1) = g(0) = 2\pi$. That is,
\[ g(1) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, dz = 2\pi. \]
Lemma IV.2.A. If $|z| < 1$ then $\int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi$.

Proof (continued).

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\[ g(1) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, dz = 2\pi. \]
**Theorem IV.2.6**

**Proposition IV.2.6.** Let $f : G \to \mathbb{C}$ be analytic and suppose $\overline{B}(a; r) \subseteq G$ ($r > 0$). If $\gamma(t) = a + re^{it}$, and $0 \leq t \leq 2\pi$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw$$

for $|z - a| < r$.

**Proof.** Without loss of generality, we assume $a = 0$ and $r = 1$ (otherwise, we consider $g(z) = f(a + rz)$ and $G_1 = \{\frac{1}{r}(z - a) \mid z \in G\}$). That is, $\overline{B}(0, 1) \subset G$. 

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\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw
\]

for \( |z - a| < r \).

Proof. Without loss of generality, we assume \( a = 0 \) and \( r = 1 \) (otherwise, we consider \( g(z) = f(a + rz) \) and \( G_1 = \{ \frac{1}{r}(z - a) \mid z \in G \} \)). That is, \( \overline{B}(0, 1) \subseteq G \). Fix \( z \) where \( |z| < 1 \). We then need to show that

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{is})e^{is} \frac{ds}{e^{is} - z}
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This is equivalent to

$$0 = \int_{0}^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z} \, ds - 2\pi f(z) = \int_{0}^{2\pi} \left( \frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) \, ds. \quad (*)$$
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Proposition IV.2.6. Let $f : G \to \mathbb{C}$ be analytic and suppose $\overline{B}(a; r) \subseteq G$ ($r > 0$). If $\gamma(t) = a + re^{it}$, and $0 \leq t \leq 2\pi$. Then

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Proof. Without loss of generality, we assume $a = 0$ and $r = 1$ (otherwise, we consider $g(z) = f(a + rz)$ and $G_1 = \{ \frac{1}{r}(z - a) \mid z \in G \}$). That is, $\overline{B}(0, 1) \subset G$. Fix $z$ where $|z| < 1$. We then need to show that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z} \, ds.$$

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Theorem IV.2.6 (continued)

Proof (continued). Let $\varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z)$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z + t(e^{is} - z)| = |z(1 - t) + te^{is}| \leq |z(1 - t)| + t \leq |1 - t| + t = 1 - t + t = 1$, then $\varphi$ is well defined ($f$ takes on values in $\overline{B}(0; 1) \subset G$) and is continuously differentiable. Let $g(t) = \int_{0}^{2\pi} \varphi(s, t) \, ds$. Then by Proposition IV.2.1, $g$ is continuously differentiable. Notice that

$$g(0) = \int_{0}^{2\pi} \varphi(s, 0) \, ds = \int_{0}^{2\pi} \left( \frac{f(z)e^{is}}{e^{is} - z} - f(z) \right) \, ds$$

$$= f(z) \int_{0}^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds - 2\pi f(z)$$

$$= 0 \text{ by Lemma IV.2.6}$$
Theorem IV.2.6 (continued)

Proof (continued). Let $\varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z)$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Since $|z + t(e^{is} - z)| = |z(1 - t) + te^{is}| \leq |z(1 - t)| + t \leq |1 - t| + t = 1 - t + t = 1$, then $\varphi$ is well defined ($f$ takes on values in $\overline{B}(0; 1) \subset G$) and is continuously differentiable. Let $g(t) = \int_0^{2\pi} \varphi(s, t) \, ds$. Then by Proposition IV.2.1, $g$ is continuously differentiable. Notice that

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$$= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds - 2\pi f(z)$$

$$= 0 \text{ by Lemma IV.2.6}$$

We now show $g$ is constant. By Proposition IV.2.1, $g'(t) = \int_0^{2\pi} \varphi_2(s, t) \, ds$ where $\varphi_2(s, t) = e^{is}f'(z + t(e^{is} - z)) = \partial \varphi / \partial t$. 


Theorem IV.2.6 (continued)

Proof (continued). Let \( \varphi(s, t) = \frac{f(z + t(e^{is} - z))e^{is}}{e^{is} - z} - f(z) \) for \( 0 \leq t \leq 1 \) and \( 0 \leq s \leq 2\pi \). Since 
\[
|z + t(e^{is} - z)| = |z(1 - t) + te^{is}| \leq |z(1 - t)| + t \leq |1 - t| + t = 1 - t + t = 1,
\]
then \( \varphi \) is well defined (\( f \) takes on values in \( \overline{B(0; 1)} \subset G \)) and is continuously differentiable. Let \( g(t) = \int_{0}^{2\pi} \varphi(s, t) \, ds \). Then by Proposition IV.2.1, \( g \) is continuously differentiable. Notice that 
\[
g(0) = \int_{0}^{2\pi} \varphi(s, 0) \, ds = \int_{0}^{2\pi} \left( \frac{f(z)e^{is}}{e^{is} - z} - f(z) \right) \, ds \\
= f(z) \int_{0}^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds - 2\pi f(z) \\
= 0 \text{ by Lemma IV.2.6}
\]
We now show \( g \) is constant. By Proposition IV.2.1, 
\[
g'(t) = \int_{0}^{2\pi} \varphi_2(s, t) \, ds \text{ where } \varphi_2(s, t) = e^{is}f'(z + t(e^{is} - z)) = \partial \varphi / \partial t.
\]
Theorem IV.2.6 (continued again)

**Proof (continued).** For $0 < t \leq 1$, we have
\[ \Phi(s) = -it^{-1}f(z + t(e^{is} - z)) \] is a primitive of $\varphi_2(s, t)$. So
\[ g'(t) = \Phi(2\pi) - \Phi(0) = 0 \] for $0 < t \leq 1$. Since $g'$ is continuous, we must have $g'(t) = 0$ for $0 \leq t \leq 1$. Therefore $g(t)$ is constant on $[0, 1]$ and $g(1) = g(0) = 0$. 
Proof (continued). For $0 < t \leq 1$, we have
\[ \Phi(s) = -it^{-1}f(z + t(e^{is} - z)) \]
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\[
g(1) = \int_0^{2\pi} \varphi(s, 1) \, ds = \int_0^{2\pi} \left( \frac{f(z + 1(e^{is} - z))e^{is}}{e^{is} - z} - f(z) \right) \, ds
\]

\[
= \int_0^{2\pi} \left( \frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) \, ds = 0.
\]

This is (*) and the result follows.
Theorem IV.2.6 (continued again)

Proof (continued). For $0 < t \leq 1$, we have
\[ \Phi(s) = -it^{-1}f(z + t(e^{is} - z)) \] is a primitive of $\varphi_2(s, t)$. So\[ g'(t) = \Phi(2\pi) - \Phi(0) = 0 \] for $0 < t \leq 1$. Since $g'$ is continuous, we must have $g'(t) = 0$ for $0 \leq t \leq 1$. Therefore $g(t)$ is constant on $[0, 1]$ and $g(1) = g(0) = 0$. That is,
\[
g(1) = \int_0^{2\pi} \varphi(s, 1) \, ds = \int_0^{2\pi} \left( \frac{f(z + 1(e^{is} - z))e^{is}}{e^{is} - z} - f(z) \right) \, ds
\]
\[
= \int_0^{2\pi} \left( \frac{f(e^{is})e^{is}}{e^{is} - z} - f(z) \right) \, ds = 0.
\]
This is (*) and the result follows.
Lemma IV.2.7. Let \( \gamma \) be a rectifiable curve in \( \mathbb{C} \) and suppose that \( F_n \) and \( F \) are continuous on \( \{\gamma\} \) If \( F \) is the uniform limit of \( F_n \) on \( \{\gamma\} \) then
\[
\int_{\gamma} F = \lim \left( \int_{\gamma} F_n \right).
\]

**Proof.** Let \( \varepsilon > 0 \); then there is \( N \in \mathbb{N} \) such that
\[
|F_n(w) - F(w)| < \varepsilon / V(\gamma)
\]
for all \( w \in \{\gamma\} \) and \( n \geq N \).
**Lemma IV.2.7.** Let $\gamma$ be a rectifiable curve in $\mathbb{C}$ and suppose that $F_n$ and $F$ are continuous on $\{\gamma\}$ If $F$ is the uniform limit of $F_n$ on $\{\gamma\}$ then

$$\int_{\gamma} F = \lim \left( \int_{\gamma} F_n \right).$$

**Proof.** Let $\varepsilon > 0$; then there is $N \in \mathbb{N}$ such that $|F_n(w) - F(w)| < \varepsilon/V(\gamma)$ for all $w \in \{\gamma\}$ and $n \geq N$. Then

$$\left| \int_{\gamma} F - \int_{\gamma} F_n \right| = \left| \int_{\gamma} (F - F_n) \right|$$

$$\leq \int_{\gamma} |F(w) - F_n(w)| \, |dw| \text{ by Proposition IV.1.17}$$

$$< \frac{\varepsilon}{V(\gamma)} V(\gamma) = \varepsilon$$

for all $n \geq N$. So $\int_{\gamma} F = \lim (\int_{\gamma} F_n).$
Lemma IV.2.7. Let $\gamma$ be a rectifiable curve in $\mathbb{C}$ and suppose that $F_n$ and $F$ are continuous on $\{\gamma\}$ If $F$ is the uniform limit of $F_n$ on $\{\gamma\}$ then

$$\int_\gamma F = \lim \left( \int_\gamma F_n \right).$$

Proof. Let $\varepsilon > 0$; then there is $N \in \mathbb{N}$ such that

$$|F_n(w) - F(w)| < \varepsilon / V(\gamma)$$

for all $w \in \{\gamma\}$ and $n \geq N$. Then

$$\left| \int_\gamma F - \int_\gamma F_n \right| = \left| \int_\gamma (F - F_n) \right|$$

$$\leq \int_\gamma |F(w) - F_n(w)| |dw|$$

by Proposition IV.1.17

$$< \frac{\varepsilon}{V(\gamma)} V(\gamma) = \varepsilon$$

for all $n \geq N$. So $\int_\gamma F = \lim(\int_\gamma F_n)$. \qed
Theorem IV.2.8. Let $f$ be analytic in $B(a; R)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$$

for $|z - a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\geq R$.

**Proof.** Let $0 < r < R$ and then $\overline{B}(a; r) \subset B(a; R)$. If $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, then by Proposition IV.2.6, $f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dz$ for $|z - a| < r$. 

Theorem IV.2.8. Let $f$ be analytic in $B(a; R)$. Then

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**Proof.** Let $0 < r < R$ and then $\overline{B}(a; r) \subset B(a; R)$. If $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, then by Proposition IV.2.6, $f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dz$ for $|z - a| < r$. For $|z - a| < r$ and $w \in \{\gamma\}$,

$$\frac{|f(w)||z - a|^n}{|w - a|^{n+1}} \leq \frac{M}{r} \left( \frac{|z - a|}{r} \right)^n$$

where $M = \max\{|f(w)| \mid |w - a| = r\}$. 
Theorem IV.2.8. Let $f$ be analytic in $B(a; R)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$$

for $|z - a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\geq R$.

Proof. Let $0 < r < R$ and then $\overline{B}(a; r) \subset B(a; R)$. If $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, then by Proposition IV.2.6, $f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$ for $|z - a| < r$. For $|z - a| < r$ and $w \in \{\gamma\}$,

$$\left| \frac{f(w)|z - a|^n}{|w - a|^{n+1}} \right| \leq \frac{M}{r} \left( \frac{|z - a|}{r} \right)^n$$

where $M = \max\{|f(w)| \mid |w - a| = r\}$.

Since $|z - a|/r < 1$, the Weierstrass $M$-Test (with $M_n = (|z - a|/r)^n$) implies that

$$\sum_{n=1}^{\infty} \frac{f(w)(z - a)^n}{(w - a)^{n+1}}$$

converges uniformly for $w \in \{\gamma\}$. 
Theorem IV.2.8. Let $f$ be analytic in $B(a; R)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$$

for $|z - a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\geq R$.

**Proof.** Let $0 < r < R$ and then $\overline{B}(a; r) \subset B(a; R)$. If $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, then by Proposition IV.2.6, $f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dz$ for $|z - a| < r$. For $|z - a| < r$ and $w \in \{\gamma\}$,

$$\frac{|f(w)||z - a|^n}{|w - a|^{n+1}} \leq \frac{M}{r} \left( \frac{|z - a|}{r} \right)^n$$

where $M = \max\{|f(w)| \mid |w - a| = r\}$. Since $|z - a|/r < 1$, the Weierstrass $M$-Test (with $M_n = (|z - a|/r)^n$) implies that

$$\sum_{n=1}^{\infty} \frac{f(w)(z - a)^n}{(w - a)^{n+1}}$$

converges uniformly for $w \in \{\gamma\}$. 


Theorem IV.2.8

Proof (continued). From the computation before Lemma IV.2.7 we have

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \] by Proposition IV.2.6

\[ = \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{w - a} \right)^n \right) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \right) (z - a)^n \] by Lemma IV.2.7.

Next set \( a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \) and we have

\[ f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n \] where the series converges if \( |z - a| < r \). By Proposition III.2.5, \( a_n = f^{(n)}(a)/n! \).
Theorem IV.2.8

Proof (continued). From the computation before Lemma IV.2.7 we have

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw \text{ by Proposition IV.2.6}
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left( \frac{z-a}{w-a} \right)^n \right) \, dw
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \right) (z-a)^n \text{ by Lemma IV.2.7.}
\]

Next set \( a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \) and we have

\[
f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where the series converges if } |z-a| < r. \text{ By Proposition III.2.5, } a_n = f^{(n)}(a)/n!.
\]

So each \( a_n \) is (1) independent of \( z \), (2) independent of \( \{\gamma\} \), and (3) independent of \( r \). Since \( r \) was chosen arbitrarily and \( |z-a| < R \), then the series representation holds for all \( z \) such that \(|z - a| < R \) and the radius of convergence of the series is at least \( R \). \( \square \)
Theorem IV.2.8

Proof (continued). From the computation before Lemma IV.2.7 we have

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \text{ by Proposition IV.2.6} \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{w - a} \right)^n \right) \]

\[ = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \right) (z - a)^n \text{ by Lemma IV.2.7.} \]

Next set \( a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \) and we have

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \] where the series converges if \( |z - a| < r \). By Proposition III.2.5, \( a_n = f^{(n)}(a)/n! \). So each \( a_n \) is (1) independent of \( z \), (2) independent of \( \{\gamma\} \), and (3) independent of \( r \). Since \( r \) was chosen arbitrarily and \( < R \), then the series representation holds for all \( z \) such that \( |z - a| < R \) and the radius of convergence of the series is at least \( R \). \( \square \)
Theorem IV.2.14. Cauchy’s Estimate. Let $f$ be analytic in $B(a; R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a; R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$ 

**Proof.** By Corollary IV.2.13, for $r < R$ we have

$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \right| \quad \text{where } \gamma(t) = a + re^{it}, \, t \in [0, 2\pi]$$

$$\leq \frac{n!}{2\pi} \int_{\gamma} \left| \frac{f(w)}{(w - a)^{n+1}} \right| \, |dw| \quad \text{by Proposition IV.1.17(b)}$$

$$\leq \frac{n!M}{2\pi} \frac{M}{r^{n+1}} (2\pi r) \quad \text{by Proposition IV.1.17(b)}$$

$$= \frac{n!M}{r^n}.$$

Now let $r \to R^-$ and the result follows.
Theorem IV.2.14

**Theorem IV.2.14. Cauchy’s Estimate.** Let $f$ be analytic in $B(a; R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a; R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

**Proof.** By Corollary IV.2.13, for $r < R$ we have

$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \right| \quad \text{where } \gamma(t) = a + re^{it}, \ t \in [0, 2\pi]$$

$$\leq \frac{n!}{2\pi} \int_{\gamma} \left| \frac{f(w)}{(w - a)^{n+1}} \right| \, |dw| \quad \text{by Proposition IV.1.17(b)}$$

$$\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} (2\pi r) \quad \text{by Proposition IV.1.17(b)}$$

$$= \frac{n!M}{r^n}.$$

Now let $r \to R^-$ and the result follows.
**Theorem IV.2.15**

**Proposition IV.2.15.** Let $f$ be analytic in $B(a; R)$ and suppose $\gamma$ is a closed rectifiable curve in $B(a; R)$. Then $f$ has a primitive in $B(a; R)$ and so $\int_\gamma f = 0$.

**Proof.** We know by Theorem IV.2.8, that an analytic function has a power series representation: $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ for $z \in B(a; R)$. Define

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - a)^{n+1} = (z - a)\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - a)^n.$$
Proposition IV.2.15. Let \( f \) be analytic in \( B(a; R) \) and suppose \( \gamma \) is a closed rectifiable curve in \( B(a; R) \). Then \( f \) has a primitive in \( B(a; R) \) and so \( \int_{\gamma} f = 0 \).

Proof. We know by Theorem IV.2.8, that an analytic function has a power series representation: \( f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n \) for \( z \in B(a : R) \). Define

\[
F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - a)^{n+1} = (z - a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - a)^n.
\]

Then, by definition, the radius of convergence of \( F \) is

\[
\frac{1}{\lim \left( \frac{a_n}{n+1} \right)^{1/n}} = \frac{\lim(n + 1)^{1/n}}{\lim(a_n)^{1/n}} = \frac{1}{\lim(a_n)^{1/n}}
\]

and so the radius of convergence of \( F \) is the same as the radius of convergence of \( f \). So \( F \) is defined on \( B(a; R) \).
**Theorem IV.2.15**

**Proposition IV.2.15.** Let $f$ be analytic in $B(a; R)$ and suppose $\gamma$ is a closed rectifiable curve in $B(a; R)$. Then $f$ has a primitive in $B(a; R)$ and so $\int_{\gamma} f = 0$.

**Proof.** We know by Theorem IV.2.8, that an analytic function has a power series representation: $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ for $z \in B(a : R)$. Define

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - a)^{n+1} = (z - a)\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - a)^n.$$

Then, by definition, the radius of convergence of $F$ is

$$\lim \left( \frac{a_n}{n+1} \right)^{1/n} = \lim (n+1)^{1/n} = 1 \lim (a_n)^{1/n}$$

and so the radius of convergence of $F$ is the same as the radius of convergence of $f$. So $F$ is defined on $B(a; R)$. Also, by Proposition III.2.5, $F'(z) = f(z)$. So $F$ is a primitive of $f$ and by Corollary IV.1.22, $\int_{\gamma} f = 0$. \qed
Proposition IV.2.15. Let $f$ be analytic in $B(a; R)$ and suppose $\gamma$ is a closed rectifiable curve in $B(a; R)$. Then $f$ has a primitive in $B(a; R)$ and so $\int_\gamma f = 0$.

Proof. We know by Theorem IV.2.8, that an analytic function has a power series representation: $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ for $z \in B(a : R)$. Define

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - a)^{n+1} = (z - a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - a)^n.$$ 

Then, by definition, the radius of convergence of $F$ is

$$\lim \left( \frac{a_n}{n+1} \right)^{1/n} = \lim \left( \frac{a_n}{n+1} \right)^{1/n} = \lim \left( \frac{a_n}{n+1} \right)^{1/n}$$

and so the radius of convergence of $F$ is the same as the radius of convergence of $f$. So $F$ is defined on $B(a; R)$. Also, by Proposition III.2.5, $F'(z) = f(z)$. So $F$ is a primitive of $f$ and by Corollary IV.1.22, $\int_\gamma f = 0$. 
