Chapter IV. Complex Integration
IV.5. Cauchy’s Theorem and Integral Formula—Proofs of Theorems
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Lemma IV.5.1. Let $\gamma$ be a rectifiable curve and suppose $\varphi$ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let

$$F_m(z) = \int_\gamma \varphi(w)(w - z)^{-m} \, dw$$

for $z \notin \{\gamma\}$. Then each $F_m$ is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Proof. Fix $a \notin \{\gamma\}$ and let $r = d(a, \{\gamma\})$. If $b \in \mathbb{C}$ satisfies $|a - b| < \delta < r/2$, then

$$F_m(a) - F_m(b) = \int_\gamma \frac{\varphi(w)}{(w - a)^m} \, dw - \int_\gamma \frac{\varphi(w)}{(w - b)^m} \, dw$$

$$= \int_\gamma \varphi(w) \left[ \frac{1}{(w - a)^m} - \frac{1}{(w - b)^m} \right] \, dw$$

$$= \int_\gamma \varphi(w) \left( \frac{1}{w - a} - \frac{1}{w - b} \right) \left[ \sum_{k=1}^{m} \frac{1}{(w - a)^{k-1}} \frac{1}{(w - b)^{m-k}} \right] \, dw$$

by algebra.
Lemma IV.5.1. Let \( \gamma \) be a rectifiable curve and suppose \( \varphi \) is a function defined and continuous on \( \{ \gamma \} \). For each \( m \geq 1 \) let 
\[
F_m(z) = \int_\gamma \varphi(w)(w - z)^{-m} \, dw
\]
for \( z \notin \{ \gamma \} \). Then each \( F_m \) is analytic on \( \mathbb{C} \setminus \{ \gamma \} \) and 
\[
F'_m(z) = m F_{m+1}(z).
\]

Proof. Fix \( a \notin \{ \gamma \} \) and let \( r = d(a, \{ \gamma \}) \). If \( b \in \mathbb{C} \) satisfies 
\[
|a - b| < \delta < r/2,
\]
then 
\[
F_m(a) - F_m(b) = \int_\gamma \frac{\varphi(w)}{(w - a)^m} \, dw - \int_\gamma \frac{\varphi(w)}{(w - b)^m} \, dw
\]
\[
= \int_\gamma \varphi(w) \left[ \frac{1}{(w - a)^m} - \frac{1}{(w - b)^m} \right] \, dw
\]
\[
= \int_\gamma \varphi(w) \left( \frac{1}{w - a} - \frac{1}{w - b} \right) \left[ \sum_{k=1}^{m} \frac{1}{(w - a)^{k-1} (w - b)^{m-k}} \right] \, dw
\]
by algebra.
Lemma IV.5.1 (continued 1)

Proof (continued).

\[
\begin{align*}
    &= \int_\gamma \varphi(w) \frac{(a - b)}{(w - a)(w - b)} \left[ \sum_{k=1}^{m} \frac{1}{(w - a)^{k-1}(w - b)^{m-k}} \right] \, dw \\
    &= \int_\gamma \varphi(w)(a - b) \left[ \sum_{k=1}^{m} \frac{1}{(w - a)^{k}(w - b)^{m-k+1}} \right] \, dw \\
    &= \int_\gamma \varphi(w)(a - b) \left[ \frac{1}{(w - a)(w - b)^{m}} + \frac{1}{(w - a)^{2}(w - b)^{m-1}} \\
    &\quad + \cdots + \frac{1}{(w - a)^{m}(w - b)} \right] \, dw. \quad (5.2)
\end{align*}
\]

(We now mimic the proof of Theorem IV.4.4.) But for \( |a - b| < r/2 \) and \( w \in \{\gamma\} \) we have that \( |w - a| \geq r > r/2 \) and \( |w - b| \geq r > r/2 \).
Lemma IV.5.1 (continued 1)

Proof (continued).

\[
\begin{align*}
= & \int_{\gamma} \varphi(w) \frac{(a-b)}{(w-a)(w-b)} \left[ \sum_{k=1}^{m} \frac{1}{(w-a)^{k-1}(w-b)^{m-k}} \right] \, dw \\
= & \int_{\gamma} \varphi(w)(a-b) \left[ \sum_{k=1}^{m} \frac{1}{(w-a)^{k}(w-b)^{m-k+1}} \right] \, dw \\
= & \int_{\gamma} \varphi(w)(a-b) \left[ \frac{1}{(w-a)(w-b)^{m}} + \frac{1}{(w-a)^{2}(w-b)^{m-1}} \\
& \quad + \cdots + \frac{1}{(w-a)^{m}(w-b)} \right] \, dw. \quad (5.2)
\end{align*}
\]

(We now mimic the proof of Theorem IV.4.4.) But for \(|a-b| < r/2\) and \(w \in \{\gamma\}\) we have that \(|w-a| \geq r > r/2\) and \(|w-b| \geq r > r/2\).
Lemma IV.5.1 (continued 2)

Proof (continued). It follows that

\[ |F_m(a) - F_m(b)| \leq |a - b| \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma) \]

\[ < \delta \max_{w \in \{\gamma\}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma). \]

So if \( \varepsilon > 0 \) is given, then by choosing \( \delta > 0 \) to be smaller than \( r/2 \) and

\[ \frac{(r/2)^{m+1}\varepsilon}{\max_{w \in \{\gamma\}} |\varphi(w)| mV(\gamma)}, \]

we see that \( F_m \) is continuous.

Fix \( a \in \mathbb{C} \setminus \{\gamma\} = G \) and \( z \in G, z \neq a \). From (5.2) (with \( b = z \)) we have

\[ \frac{F_m(a) - F_m(z)}{a - z} = \int_\gamma \frac{\varphi(w)}{(w - a)(w - z)^m} dw + \int_\gamma \frac{\varphi(w)}{(w - a)^2(w - z)^{m-1}} dw + \cdots + \int_\gamma \frac{\varphi(w)}{(w - a)^m(w - z)} dw. \]
Lemma IV.5.1 (continued 2)

Proof (continued). It follows that

$$|F_m(a) - F_m(b)| \leq |a - b| \max_{w \in \{ \gamma \}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma)$$

$$< \delta \max_{w \in \{ \gamma \}} |\varphi(w)| \frac{m}{(r/2)^{m+1}} V(\gamma).$$

So if $\varepsilon > 0$ is given, then by choosing $\delta > 0$ to be smaller than $r/2$ and

$$\frac{(r/2)^{m+1}\varepsilon}{\max_{w \in \{ \gamma \}} |\varphi(w)| m V(\gamma)},$$

we see that $F_m$ is continuous.

Fix $a \in \mathbb{C} \setminus \{ \gamma \} = G$ and $z \in G$, $z \neq a$. From (5.2) (with $b = z$) we have

$$\frac{F_m(a) - F_m(z)}{a - z} = \int_{\gamma} \frac{\varphi(w)}{(w - a)(w - z)^m} \, dw + \int_{\gamma} \frac{\varphi(w)}{(w - a)^2(w - z)^{m-1}} \, dw + \cdots + \int_{\gamma} \frac{\varphi(w)}{(w - a)^m(w - z)} \, dw.$$
Lemma IV.5.1. Let $\gamma$ be a rectifiable curve and suppose $\phi$ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z) = \int_{\gamma} \phi(w)(w - z)^{-m} \, dw$ for $z \notin \{\gamma\}$. Then each $F_m$ is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F'_m(z) = mF_{m+1}(z)$.

Proof (continued). By the first part of the proof, each integral on the right hand side is a continuous function of $z$ ($z$ has replaced $b$ in the new notation; to apply the continuity from above, we can let $\phi(w)$ absorb the power of $w - a$ so that each integral is in the form addressed above) for $z \in G = \mathbb{C} \setminus \{\gamma\}$. So with $z \to a$ we have

$$F'_m(a) = m \int_{\gamma} \frac{\phi(w)}{(w - a)^{m+1}} \, dw = mF_{m+1}(a).$$

Since $a \notin \{\gamma\}$ is arbitrary, the result follows.
Theorem IV.5.4

Theorem IV.5.4. Cauchy’s Integral Formula (First Version).
Let $G$ be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma$ is a closed rectifiable curve in $G$ such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz.$$ 

Proof. Define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z, w) = \frac{f(z) - f(w)}{z - w}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Then $\varphi$ is continuous and for each $w \in G$, $z \to \varphi(z, w)$ is analytic (by Exercise IV.5.1).
Theorem IV.5.4. Cauchy’s Integral Formula (First Version).
Let $G$ be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma$ is a closed rectifiable curve in $G$ such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \, dz.$$ 

**Proof.** Define $\varphi : G \times G \to \mathbb{C}$ by $\varphi(z, w) = \frac{f(z) - f(w)}{z - w}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Then $\varphi$ is continuous and for each $w \in G$, $z \to \varphi(z, w)$ is analytic (by Exercise IV.5.1). Let $H = \{w \in \mathbb{C} \mid n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is continuous and integer-valued on components of $G \setminus \{\gamma\}$ (by Theorem IV.4.4), $H$ is open. Moreover, $H \cup G = \mathbb{C}$ since $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$. 


Theorem IV.5.4. Cauchy’s Integral Formula (First Version).

Let $G$ be an open subset of the plane and $f : G \rightarrow \mathbb{C}$ an analytic function. If $\gamma$ is a closed rectifiable curve in $G$ such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz.$$  

**Proof.** Define $\varphi : G \times G \rightarrow \mathbb{C}$ by $\varphi(z, w) = \frac{f(z) - f(w)}{z-w}$ if $z \neq w$ and $\varphi(z, z) = f'(z)$. Then $\varphi$ is continuous and for each $w \in G$, $z \rightarrow \varphi(z, w)$ is analytic (by Exercise IV.5.1). Let $H = \{w \in \mathbb{C} \mid n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is continuous and integer-valued on components of $G \setminus \{\gamma\}$ (by Theorem IV.4.4), $H$ is open. Moreover, $H \cup G = \mathbb{C}$ since $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$. 

Proof (continued). Define \( g : \mathbb{C} \to \mathbb{C} \) as \( g(z) = \int_\gamma \varphi(z, w) \, dw \) if \( z \in G \) and \( g(z) = \int_\gamma \frac{f(w)}{w-z} \, dw \) if \( z \in H \). We need to make sure this piecewise definition is consistent for \( z \in G \cap H \). If \( z \in G \cap H \) then

\[
\int_\gamma \varphi(z, w) \, dw = \int_\gamma \frac{f(w) - f(z)}{w-z} \, dw
\]

\[
= \int_\gamma \frac{f(w)}{w-z} \, dw - f(z) \int_\gamma \frac{1}{w-z} \, dw
\]

\[
= \int_\gamma \frac{f(w)}{w-z} \, dw - f(z) n(\gamma; z) \times 2\pi i
\]

\[
= \int_\gamma \frac{f(w)}{w-z} \, dw \text{ since } n(\gamma; z) = 0 \text{ and } z \in H.
\]

Hence, \( G \) is well-defined.
Proof (continued). Define $g : \mathbb{C} \to \mathbb{C}$ as $g(z) = \int_\gamma \varphi(z, w) \, dw$ if $z \in G$ and $g(z) = \int_\gamma \frac{f(w)}{w-z} \, dw$ if $z \in H$. We need to make sure this piecewise definition is consistent for $z \in G \cap H$. If $z \in G \cap H$ then

$$\int_\gamma \varphi(z, w) \, dw = \int_\gamma \frac{f(w) - f(z)}{w-z} \, dw = \int_\gamma \frac{f(w)}{w-z} \, dw - f(z) \int_\gamma \frac{1}{w-z} \, dw = \int_\gamma \frac{f(w)}{w-z} \, dw - f(z)n(\gamma; z) \times 2\pi i = \int_\gamma \frac{f(w)}{w-z} \, dw \text{ since } n(\gamma; z) = 0 \text{ and } z \in H.$$

Hence, $G$ is well-defined.
Theorem IV.5.4 (continued 2)

Proof (continued). For \( z \in G \), \( g(z) \) is analytic by Lemma IV.5.1 with \( m = 1 \) and numerator \( f(z) - f(w) \). For \( z \in H \), \( g(z) \) is analytic by Lemma IV.5.1 with \( m = 1 \) and numerator \( f(w) \). So \( g \) is an entire function. By Theorem IV.4.4, \( H \) contains a neighborhood of \( \infty \) in \( \mathbb{C}_\infty \). Since \( f \) is bounded on \( \{\gamma\} \) and \( \lim_{z \to \infty} \frac{1}{w - z} = 0 \) uniformly for \( w \in \{\gamma\} \) (both follow since \( \{\gamma\} \) is compact), we have

\[
\lim_{z \to \infty} g(z) = \lim_{z \to \infty} \int_{\gamma} \frac{f(w)}{w - z} \, dw \quad \text{since for } z \text{ sufficiently large, } z \in H
\]

\[
= \int_{\gamma} \left( \lim_{z \to \infty} \frac{f(w)}{w - z} \right) \, dw \quad \text{by the uniform convergence}
\]

\[
= \int_{\gamma} f(w) \lim_{z \to \infty} \frac{1}{w - z} \, dw
\]

\[
= 0 \quad \text{since } f(w) \text{ is bounded on } \gamma.
\]
Proof (continued). For $z \in G$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(z) - f(w)$. For $z \in H$, $g(z)$ is analytic by Lemma IV.5.1 with $m = 1$ and numerator $f(w)$. So $g$ is an entire function. By Theorem IV.4.4, $H$ contains a neighborhood of $\infty$ in $C_{\infty}$. Since $f$ is bounded on $\{\gamma\}$ and $\lim_{z \to \infty} 1/(w - z) = 0$ uniformly for $w \in \{\gamma\}$ (both follow since $\{\gamma\}$ is compact), we have

$$\lim_{z \to \infty} g(z) = \lim_{z \to \infty} \int_{\gamma} \frac{f(w)}{w - z} \, dw$$

since for $z$ sufficiently large, $z \in H$

$$= \int_{\gamma} \left( \lim_{z \to \infty} \frac{f(w)}{w - z} \right) \, dw \text{ by the uniform convergence}$$

$$= \int_{\gamma} f(w) \lim_{z \to \infty} \frac{1}{w - z} \, dw$$

$$= 0 \text{ since } f(w) \text{ is bounded on } \gamma.$$
Theorem IV.5.4 (continued 3)

**Proof (continued).** So there exists $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$ (i.e., $z \in \mathbb{C} \setminus B(0; R)$). However, $g$ is bounded on $\overline{B}(0; R)$ (since $g$ is continuous and $\overline{B}(0; R)$ is compact). But then, $g$ is a bounded entire function. So by Liouville’s Theorem, $g$ is constant. In fact, $g \equiv 0$ since $\lim_{z \to \infty} g(z) = 0$. So for $a \in G \setminus \{\gamma\}$,

$$
0 = g(a) = \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz \quad \text{since } a \in G \ (w \text{ replaced with } z)
$$

$$
= \int_{\gamma} \frac{f(z)}{z - a} \, dz - f(a) \int_{\gamma} \frac{1}{z - a} \, dz
$$

$$
= \int_{\gamma} \frac{f(z)}{z - a} \, dz - f(a) n(\gamma; a) 2\pi i.
$$

So,

$$
n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \, dz.
$$
Theorem IV.5.4 (continued 3)

Proof (continued). So there exists $R > 0$ such that $|g(z)| \leq 1$ for $|z| \geq R$ (i.e., $z \in \mathbb{C} \setminus B(0; R)$). However, $g$ is bounded on $\overline{B}(0; R)$ (since $g$ is continuous and $\overline{B}(0; R)$ is compact). But then, $g$ is a bounded entire function. So by Liouville’s Theorem, $g$ is constant. In fact, $g \equiv 0$ since $\lim_{z \to \infty} g(z) = 0$. So for $a \in G \setminus \{\gamma\}$,

$$0 = g(a) = \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz \text{ since } a \in G \text{ (w replaced with } z)$$

$$= \int_{\gamma} \frac{f(z)}{z - a} \, dz - f(a) \int_{\gamma} \frac{1}{z - a} \, dz$$

$$= \int_{\gamma} \frac{f(z)}{z - a} \, dz - f(a)n(\gamma; a)2\pi i.$$

So,

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \, dz.$$
Theorem IV.5.8

**Theorem IV.5.8.** Let $G$ be an open set and $f : G \rightarrow \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in $G$ such that 

$$n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0$$

for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$ and $k \geq 1$,

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \left( \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}} \, dz \right).$$

**Proof.** Differentiate $k$ times the conclusion of Theorem IV.5.6 with respect to $a$:

$$\frac{d^k}{da^k} \left[ f(a) \sum_{j=1}^{m} n(\gamma_j; a) \right] = \frac{d^k}{da^k} \left[ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} \, dz \right].$$

Since $\sum_{j=1}^{m} n(\gamma_j; a)$ is constant and by repeated application of IV.5.1, the claim follows.
Theorem IV.5.8

**Theorem IV.5.8.** Let $G$ be an open set and $f : G \to \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in $G$ such that

$$n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0 \text{ for all } w \in \mathbb{C} \setminus G$$

then for $a \in G \setminus \{\gamma\}$ and $k \geq 1$,

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \left( \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z - a)^{k+1}} \, dz \right).$$

**Proof.** Differentiate $k$ times the conclusion of Theorem IV.5.6 with respect to $a$:

$$\frac{d^k}{da^k} \left[ f(a) \sum_{j=1}^{m} n(\gamma_j; a) \right] = \frac{d^k}{da^k} \left[ \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{k+1}} \, dz \right].$$

Since $\sum_{j=1}^{m} n(\gamma_j; a)$ is constant and by repeated application of IV.5.1, the claim follows. \qed
Exercise IV.5.5. Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_{\gamma} (z - a)^{-n} \, dz = 0$.

Solution. Define $f(z) \equiv 1$ and $k = n - 1$. Applying Theorem IV.5.8 (with $m = 1$) gives

$$f^{(n-1)}(a)n(\gamma; a) = (n - 1)! \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^n} \, dz,$$

or $0 = \int_{\gamma} \frac{f(z)}{(z - a)^n} \, dz$ (since $f^{(n-1)}(a) = 0$). \hfill $\Box$
Exercise IV.5.5. Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_{\gamma} (z - a)^{-n} \, dz = 0$.

Solution. Define $f(z) \equiv 1$ and $k = n - 1$. Applying Theorem IV.5.8 (with $m = 1$) gives

$$f^{(n-1)}(a)n(\gamma; a) = (n-1)! \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^n} \, dz,$$

or $0 = \int_{\gamma} \frac{f(z)}{(z-a)^n} \, dz$ (since $f^{(n-1)}(a) = 0$). \qed
Example. Compute \( \int_{|z|=1} e^z z^{-n} \, dz \). (This is from page 123 of Lars Ahlfors *Complex Analysis*).

Solution. Here, we take \( f(z) = e^z \), \( a = 0 \), \( k = n - 1 \), and \( \gamma(t) = e^{it} \), \( t \in [0, 2\pi] \). Then by Corollary IV.5.9,

\[
\begin{align*}
&f^{(k)}(a)n(\gamma; a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{k+1}} \, dz \text{ implies} \\
&f^{(n-1)}(0)n(\gamma; 0) = \frac{(n - 1)!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - 0)^n} \, dz \text{ or} \\
(e^0)(1) = \frac{(n - 1)!}{2\pi i} \int_{\gamma} \frac{e^z}{z^n} \, dz. \text{ So } \int_{\gamma} e^z z^{-n} \, dz = \frac{2\pi i}{(n - 1)!}. 
\end{align*}
\]
Example. Compute \( \int_{\{|z|=1\}} e^z z^{-n} \, dz \). (This is from page 123 of Lars Ahlfors Complex Analysis).

Solution. Here, we take \( f(z) = e^z \), \( a = 0 \), \( k = n - 1 \), and \( \gamma(t) = e^{it} \), \( t \in [0, 2\pi] \). Then by Corollary IV.5.9,

\[
\begin{align*}
  f^{(k)}(a)n(\gamma; a) & = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{k+1}} \, dz \quad \text{implies} \\
  f^{(n-1)}(0)n(\gamma; 0) & = \frac{(n-1)!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - 0)^n} \, dz \quad \text{or} \\
  (e^0)(1) & = \frac{(n-1)!}{2\pi i} \int_{\gamma} \frac{e^z}{z^n} \, dz \quad \text{So} \quad \int_{\gamma} e^z z^{-n} \, dz = \frac{2\pi i}{(n-1)!}.
\end{align*}
\]
Theorem IV.5.10

Theorem IV.5.10. Morera’s Theorem.
Let $G$ be a region and let $f : G \to \mathbb{C}$ be a continuous function such that
\[ \int_T f = 0 \]
for every closed triangular path $T$ in $G$ (i.e., $T$ is a closed polygon with 3 sides); then $f$ is analytic in $G$.

Proof. Without loss of generality, we assume $G = B(a; R)$ (otherwise, we can write $G$ as a union of disks). We show that $f$ has a primitive $F$ and then we know $F$ is analytic and hence so is $F' = f$. For $z \in G$, define $F(z) = \int_{[a,z]} f(z) \, dz$. Fix $z_0 \in G$. Then for any $z \in G$, by hypothesis (since $a$, $z$, and $z_0$ form a triangle in $G$),
\[
F(z) = \int_{[a,z]} f(z) \, dz = \int_{[a,z_0]} f(z) \, dz + \int_{[z_0,z]} f(z) \, dz.
\]
Hence,
\[
\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f(z) \, dz.
\]
Theorem IV.5.10.

**Morera’s Theorem.**

Let $G$ be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path $T$ in $G$ (i.e., $T$ is a closed polygon with 3 sides); then $f$ is analytic in $G$.

**Proof.** Without loss of generality, we assume $G = B(a; R)$ (otherwise, we can write $G$ as a union of disks). We show that $f$ has a primitive $F$ and then we know $F$ is analytic and hence so is $F' = f$. For $z \in G$, define $F(z) = \int_{[a, z]} f(z) \, dz$. Fix $z_0 \in G$. Then for any $z \in G$, by hypothesis (since $a$, $z$, and $z_0$ form a triangle in $G$),

$$F(z) = \int_{[a,z]} f(z) \, dz = \int_{[a,z_0]} f(z) \, dz + \int_{[z_0,z]} f(z) \, dz.$$ 

Hence,

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f(z) \, dz.$$
Theorem IV.5.10 (continued 1)

Proof (continued). This gives

\[
\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} (f(z) - f(z_0)) \, dz
\]

\[
= \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) \, dw.
\]

So

\[
\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = \left| \frac{1}{z - z_0} \int_{[z_0, z]} (f(w) - f(z_0)) \, dw \right|
\]

\[
\leq \frac{1}{|z - z_0|} \int_{[z_0, z]} |f(w) - f(z_0)| \, |dw|
\]

\[
\leq \frac{|z - z_0|}{|z - z_0|} \sup \{|f(z) - f(z_0)| \mid w \in [z, z_0]\}
\]

\[
= \sup \{|f(w) - f(z_0)| \mid w \in [z, z_0]\}.
\]
**Theorem IV.5.10. Morera’s Theorem.**
Let $G$ be a region and let $f : G \rightarrow \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path $T$ in $G$ (i.e., $T$ is a closed polygon with 3 sides); then $f$ is analytic in $G$.

**Proof (continued).** Since $f$ is continuous,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

So $F$ is analytic and hence $f = F'$ is analytic.