Theorem V.2.2

Residue Theorem.
Let $f$ be analytic in the region $G$, except for the isolated singularities $a_1, a_2, \ldots, a_m$. If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any of the points $a_k$ and if $\gamma \approx 0$ in $G$ then
\[
\frac{1}{2\pi i} \int_\gamma f = \sum_{k=1}^m n(\gamma; a_k) \text{Res}(f; a_k).
\]

Proof. Define $m_k = n(\gamma; a_k)$ for $1 \leq k \leq m$. Choose positive $r_1, r_2, \ldots, r_m$ such that the discs $B(a_i; r_i)$ are disjoint, none of them intersect $\{\gamma\}$, and each disc is contained in $G$. This can be done since $\{\gamma\}$ is compact (by Theorem II.5.17) and $G$ is open. Let $\gamma_k(t) = a_k + r_k \exp(-2\pi im_k t)$ for $0 \leq t \leq 1$.

Theorem V.2.2 (continued 2)

Proof (continued). So the uniform convergence gives
\[
\int_{\gamma_k} f(z) \, dz = \int_{\gamma_k} \left( \sum_{n=-\infty}^{\infty} b_n (z - a_k)^n \right) = \sum_{n=-\infty}^{\infty} b_n \left( \int_{\gamma_k} (z - a_k)^n \, dz \right).
\]
Now for $n \neq -1$, $(z - a_k)^n$ has a primitive and $\int_{\gamma_k} (z - a_k)^n \, dz = 0$. When $n = -1$,
\[
b_{-1} \int_{\gamma_k} (z - a_k)^{-1} \, dz = \text{Res}(f; a_k) \int_{\gamma_k} (z - a_k)^{-1} \, dz
\]
by the definition of residue
\[
= \text{Res}(f; a_k) 2\pi i \text{Res}(f; a_k)
\]
by the definition of winding number.
Theorem V.2.2. Residue Theorem.
Let \( f \) be analytic in the region \( G \), except for the isolated singularities \( a_1, a_2, \ldots, a_m \). If \( \gamma \) is a closed rectifiable curve in \( G \) which does not pass through any of the points \( a_k \) and if \( \gamma \approx 0 \) in \( G \) then
\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \sum_{k=1}^{m} n(\gamma; a_k) \text{Res}(f; a_k).
\]

Proof (continued). So (2.3) gives that
\[
\int_{\gamma} f(z) \, dz = -\sum_{k=1}^{m} \left( \int_{\gamma_k} f(z) \, dz \right)
\]
\[
= -\sum_{k=1}^{m} 2\pi i n(\gamma_k; a_k) \text{Res}(f; a_k) = 2\pi i \sum_{k=1}^{m} n(\gamma; a_k) \text{Res}(f; a_k)
\]

since \( n(\gamma_k; a_k) = -n(\gamma; a_k) \). Therefore,
\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \sum_{k=1}^{m} n(\gamma; a_k) \text{Res}(f; a_k). \quad \square
\]

Proposition V.2.4. Suppose \( f \) has a pole of order \( m \) at \( z = a \). Let \( g(z) = (z - a)^m f(z) \). Then
\[
\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).
\]

Proof. By Proposition V.1.4 and the definition of “pole of order \( m \),” we have that \( g(z) \) has a removable singularity at \( z = a \) and \( g(a) = b_0 \neq 0 \) (here, we technically mean that \( \lim_{z \to a} g(z) = b_0 \neq 0 \)). Let \( g(z) = \sum_{k=1}^{\infty} b_k (z-a)^k \) be the power series of \( g \) about \( z = a \). Then for \( z \) “near” \( a \) but not equal to \( a \), we have
\[
f(z) = \frac{b_0}{(z-a)^m} + \frac{b_1}{(z-a)^{m-1}} + \cdots + \frac{b_{m-1}}{z-a} + \sum_{k=0}^{\infty} b_{m+k}(z-a)^k.
\]

So this is the Laurent series of \( f \) about \( z = a \), and so \( \text{Res}(f; a) = b_{m-1} \). Since \( b_{m-1} \) is the coefficient for \( (z-a)^{m-1} \) is the power series representation of \( g \), so \( b_{m-1} = g^{(m-1)}(a)/(m-1)! \). \quad \square