Proposition 3.7. Let $f$ be analytic on an open set containing $\mathbb{B}$. Then for each $\gamma \in \mathbb{C}$, the circle $\gamma = |\gamma| \neq 0$, the function $f$ is analytic and has a pole at each zero of $\gamma$. Suppose that $f$ is analytic on $\mathbb{B}$. Then for each $\gamma \in \mathbb{C}$, the circle $\gamma = |\gamma| \neq 0$, the function $f$ is analytic and has a pole at each zero of $\gamma$.

Proof. By repeated application of (3.7) to (3.8), we have

$$\frac{(z)f}{(z)g} + \sum_{i=1}^{n} \frac{f(z) - z}{(z)g} + \sum_{j=1}^{m} \frac{z - z}{(z)f} = \frac{(z)f}{(z)g}$$

and each pole we have (z:3.8) and (3.7) to each zero

$$\sum_{i=1}^{n} \frac{f(z) - z}{(z)g} + \sum_{j=1}^{m} \frac{z - z}{(z)f} = \frac{(z)f}{(z)g}$$

Theorem 3.4. Argument Principle. Suppose that $f$ is an analytic function on a simply connected domain $G$ and let $\gamma$ be a closed curve in $G$ which is not a closed rectifiable contour in $G$. Then

$$\int_{\gamma} \frac{f'}{f} = 2\pi i \sum \text{Res}$$

where $\text{Res}$ is the residue of $f$ at each zero of $f$.
\[ (z^d - z) - (z^d - z) = \left( \int \frac{z^d}{z} \, dz - \int \frac{z^d}{z} \, dz \right) = \left( \int \frac{z^d}{z} \, dz \right) \cdot \left( 1 - \frac{1}{z} \right) = \int z^d \cdot \frac{1}{z} \, dz = \frac{1}{z} \int z^d \, dz \]

By the Argument Principle.

Proof (continued).

So \( z^d - z = z^d - z \). Then \( \left| (z^d) \right| + \left| (z) \right| > \left| (z^d) \right| \) for some \( z \in \mathbb{C} \) and \( \Re z > 0 \).

By hypothesis.

Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( (\mathbb{C}, \mathbb{C}) \).

Theorem 3.8: Rouche's Theorem.