Complex Analysis 1, MATH 5510, Spring 2022

Homework 12, Section IV.3

Due Saturday, April 23

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

IV.3.1. Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Prove that f is a polynomial of degree $\leq n$. HINT: Consider $|f(z)|$ on $|z| = r$ for $r > R$. Use the Maximum Modulus Theorem—Second Version to get a bound for $|f(z)|$ on $B(0;r)$. Use Cauchy's Estimate to show $f^{(k)}(0) = 0$ for $k > n$ (similar to the proof of Liouville's Theorem).

IV.3.6. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $a \in G$ such that $|f(a)| \leq |f(z)|$ for all $z \in G$. Prove that either $f(a) = 0$ or f is constant. HINT: Consider $1/f$; apply the Maximum Modulus Theorem.

IV.3.8. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all $z \in G$. Prove that either $f \equiv 0$ or $g \equiv 0$ on G. HINT: Consider the zeros of f and of g in some $\overline{B}(a,r) \subset G$. Apply Theorem IV.3.7.