Complex Analysis 1, MATH 5510, Fall 2017

Homework 3, Section II.1 (Revised)

Due: Tuesday, September 19 at 2:15

II.1.1 (b) (1.3) Let $X = \mathbb{C}$ and define $d(x+iy, a+ib) = |x-a| + |y-b|$. Then $(\mathbb{C}, d)$ is a metric space.

II.1.1. (e) (1.6) Let $X = \mathbb{R}^n$ and for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ define $d(x,y) = \left\{\sum_{j=1}^n (x_j - y_j)^2)\right\}^{1/2}$. Then $(\mathbb{R}^n, d)$ is a metric space. HINT: You may assume the Schwarz Inequality from Linear Algebra: $|\vec{x} \cdot \vec{y}| = \left|\sum_{j=1}^n x_j y_j\right| \leq \|\vec{x}\|\|\vec{y}\| = \left\{\sum_{j=1}^n x_j^2\right\}^{1/2} \left\{\sum_{j=1}^n y_j^2\right\}^{1/2}$.

II.1.2. (b) Is the real axis $A = \{z \mid \operatorname{Im}(z) = 0\} \subset \mathbb{C}$ is open or closed?

II.1.5. Let (X, d) be a metric space.

(a) Prove that the sets X and $\varnothing$ are closed.

(b) Prove that if $F_1, F_2, \ldots, F_n$ are closed sets in X then $\cup_{k=1}^n F_k$ is closed.

(c) Prove that if $\{F_j \mid j \in J\}$ is any countable collection of closed sets in X where J is any indexing set (possibly uncountable) then $F = \cap\{F_j \mid j \in J\}$ is also closed.

II.1.7. BONUS. Consider $(\mathbb{C}_\infty, d)$ where

$$\begin{aligned} d(z, z') &= \frac{2|z-z'|}{((1+|z|^2)(1+|z'|^2))^{1/2}} \text{ for } z, z' \in \mathbb{C} \\ d(z, \infty) &= d(\infty, z) = \frac{2}{(1+|z|^2)^{1/2}} \text{ for } z \in \mathbb{C} \end{aligned}$$

and $d(\infty, \infty) = 0$. It is easy to show that $d(z_1, z_2) \geq 0$, $d(z_1, z_2) = 0$ if and only if $z_1 = z_2$, and $d(z_1, z_2) = d(z_2, z_1)$ for all $z_1, z_2 \in \mathbb{C}_\infty$. Prove that d satisfies the Triangle Inequality. You must show

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C},$$

$$d(z_1, z_2) \leq d(z_1, \infty) + d(\infty, z_2) \text{ for all } z_1, z_2 \in \mathbb{C}, \text{ and}$$

$$d(z_1, \infty) \leq d(z_1, z_2) + d(z_2, \infty) \text{ for all } z_1, z_2, \in \mathbb{C}.$$

Hint: Consider the Cauchy-Schwarz Inequality on $\mathbb{C}^n$: For all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$,

$$\left|\sum_{k=1}^n x_k \overline{y}_k\right| \leq \left(\sum_{k=1}^n |x_k|^2\right)\left(\sum_{k=1}^n |y_k|^2\right).$$

Use Cauchy-Schwarz to show that $|1 + z_1\overline{z}_2| \leq (1+|z_1|^2)(1+|z_2|^2)$ and use this in your computations.