Supplement. Hyperbolic Geometry and the Poincare Disk

Note. Let's work our way back to geometry. The following definition is from *Visual Complex Analysis* by Tristan Needham, Oxford University Press (1997), page 176.

Definition. An *automorphism* of a region of the complex plane is a one to one conformal mapping of the region onto itself.

Note. We are interested in automorphisms of the unit disk. We know that certain Möbius transformations will be some of these automorphisms.

Conway's Exercise III.3.10. Let $D = \{z \mid |z| < 1\}$ and find all Möbius transformations T such that $T(D) = D$.

Solution. The transformations must be of the form $T(z) = e^{i\theta}\left(\dfrac{z-\alpha}{\overline{\alpha}z-1}\right)$ where $|\alpha| < 1$.

Note. In fact, the only automorphisms of the unit disc are those given above [Needham, page 357].

Note. With all this automorphism stuff, groups can't be far behind. The collection of all Möbius transformations form a group call $\text{Aut}(\mathbb{C}_\infty)$. The collection of Möbious transformations mapping D to D are a subgroup of $\text{Aut}(\mathbb{C}_\infty)$—let's denote it $\text{Aut}(D)$.

Note. We now explore the *Poincare disk model* in detail based on Chapter 5 of *Geometry with an Introduction to Cosmic Topology* by Michael Hitchman, Boston: Jones and Bartlett Publishers (2009). When quoting result's from Hitchman, we present the number of the result and add a prefix of 'H.' We call the automorphism group of the unit disk the *hyperbolic transformation group* and denote it $\mathcal{H}$.

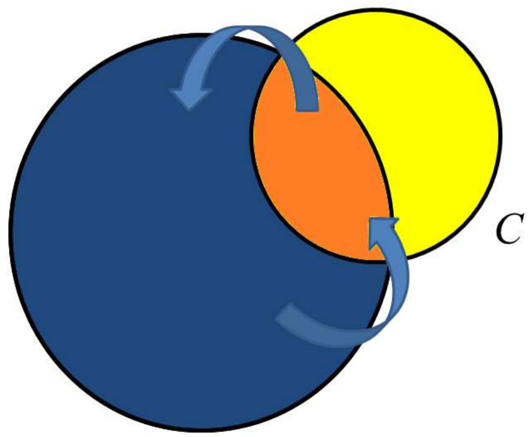
Definition H.5.1.1. The *Poincare disk model for hyperbolic geometry* is the geometry $(D, \mathcal{H})$ where D is the open unit disk.

Note. The unit circle is not part of the Poincare disk, but it still plays an important role. It is called the *circle at infinity*, denoted S^1_∞.

Recall. Corollary H.3.2.6. Let C be a circle with center z_0. Inversion in C takes clines orthogonal to C to themselves (i.e., such clines are fixed by the inversion).

Also: **Exercise H.3.2.10.** Suppose C and D are orthogonal circles (i.e., they intersect at two points and the intersection are at right angles). Then inversion in C takes the interior of D to itself.

These two results combine to give the following picture:



So if D is the unit disk, then inversion with respect to C maps D to itself and hence is an element of the hyperbolic transformation group. So inversion with respect to such clines is fundamental. Notice that if C is a line intersecting the unit circle at a right angle, then it also maps D to itself.

Definition H.5.1.2. A *hyperbolic reflection* of D is an inversion in a cline that meets S^1_∞ at right angles.

Theorem H.5.1.3. Any map in $\mathcal{H}$ is the composition of two hyperbolic reflections.

Corollary H.5.1.4. Given z_0 in D and z_1 on S^1_∞, there exists a transformation in $\mathcal{H}$ that sends z_0 to 0 and z_1 to 1.

Proof. This is the transformation T constructed in the proof of Theorem H.5.1.3. $\blacksquare$

Note. Since any map in $\mathcal{H}$ is a composition of two inversions about clines which are orthogonal to S^1_∞, we can classify the elements of $\mathcal{H}$ according to whether these clines intersect twice, once, or not at all.

Note. If the two clines intersect in D at a point p then they also intersect at p^* (this follows from the fact that the clines intersect S^1_∞ at right angles).

Definition. An element of $\mathcal{H}$ which is a composition of two inversions about clines which intersect in two points (p and p^*) is a *hyperbolic rotation* about point p.

Note. Let T represent such a hyperbolic rotation. Of course p is fixed under T (since the two clines involved both pass trough p and inversion with respect to a cline fixes the cline). T moves points along circles orthogonal to both clines:

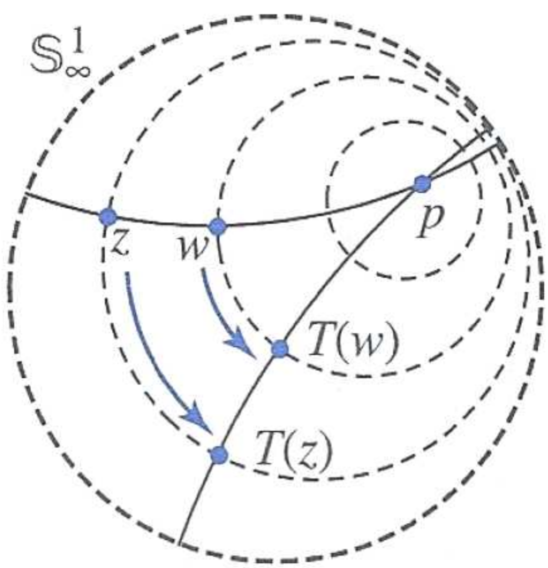


Figure 5.3b from Hitchman.

See Section 3.5 of Hitchman for details.

Note. If the two clines involved in the transformation intersect at one point, then the point must be on S^1_∞ (otherwise the clines would also intersect at p^*).

Definition. An element of $\mathcal{H}$ which is a composition of two inversions about clines which intersect in one point p (on S^1_∞) is a *parallel displacement*.

Note. In parallel displacement, points are moved along circles which pass through p and are tangent to S^1_∞ (and hence are perpendicular to the two clines determining $\mathcal{H}$). Such circles are called *horocycles*:

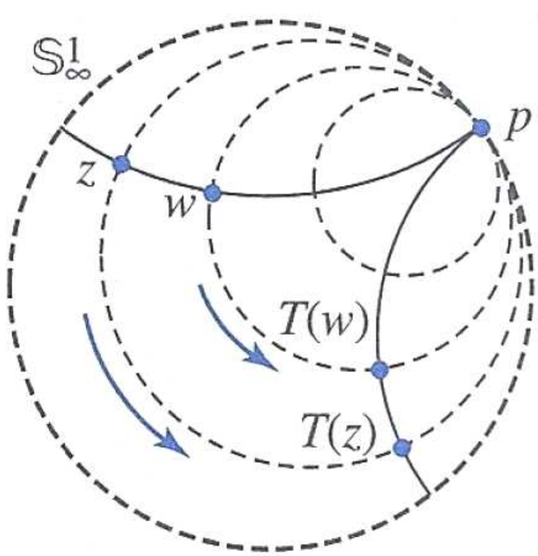


Figure 5.3b from Hitchman.

Note. If the two clines do not intersect (in which case at least one must be a circle), then there are two points p and q symmetric with respect to both clines and on S^1_∞ (see Section 3.2 of Hitchman for more details). Such a transformation moves points along arcs of circles passing through p and q:

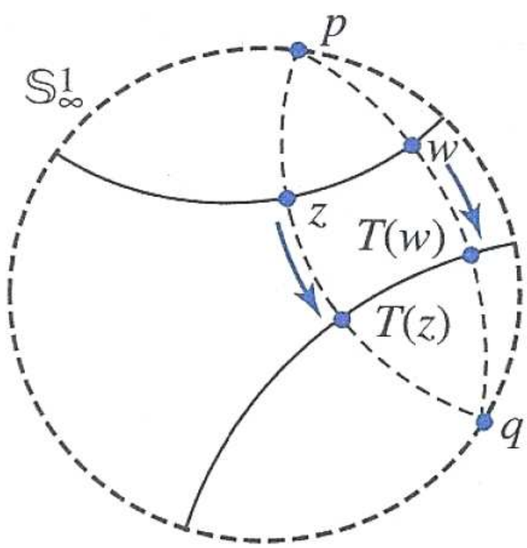
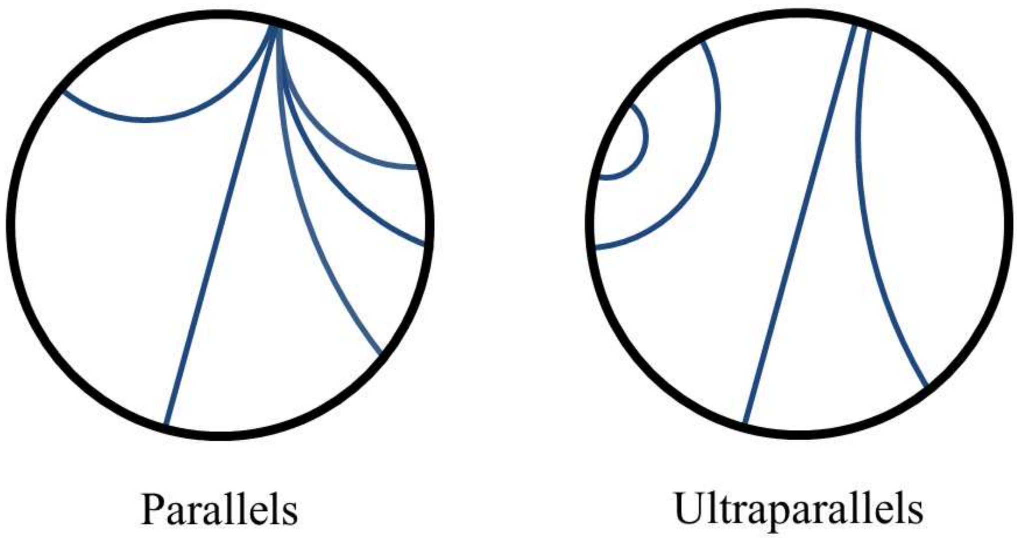


Figure 5.3c from Hitchman.

Definition. An element of $\mathcal{H}$ which is a composition of two inversions about clines which do not intersect is a *hyperbolic translation*.

Exercise H.5.1.5. Prove that any two horocycles are congruent. HINT: Rotate one circle to the other using the corresponding points "p," then use a hyperbolic translation.

Definition H.5.2.1. A *hyperbolic line* in $(D, \mathcal{H})$ is the portion of a cline inside D that intersects the unit circle at right angles. A point on the circle at infinity S^1_∞ is called an *ideal point*. [Here come a couple of odd definitions.] Two hyperbolic lines are *parallel* if they share one ideal point. [The following is not from Hitchman.] Two hyperbolic lines that do not intersect and do not share an ideal point are *ultraparallel*.



Theorem H.5.2.2. Any two hyperbolic lines are congruent in hyperbolic geometry.

Theorem H.5.2.3. Given a point z_0 and a hyperbolic line L not through z_0, there exists two distinct hyperbolic lines through z_0 that are parallel to L.

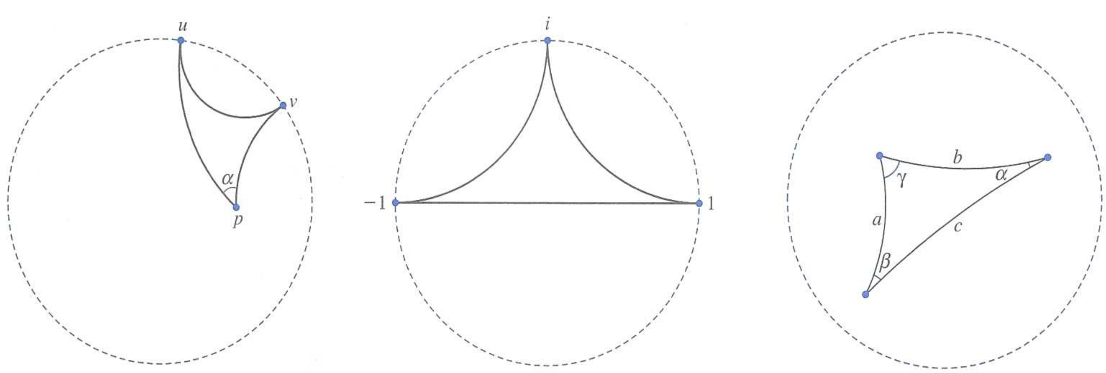
Note. Theorem H.5.2.3 certainly violates Playfair's Theorem (which is equivalent to the Parallel Postulate). So we are justified in calling hyperbolic geometry $(D, \mathcal{H})$ *non-Euclidean*. Notice that there are an infinite number of lines through $z - 0$ ultraparallel to L—these lines are wedged between the lines through u and v given in the proof.

Note. We can use three pairwise non-ultraparallel lines to define a (hyperbolic) triangle. Once we have a metric, we can measure distances and areas. Some results for triangles are:

Theorem H.5.4.6. The area of a triangle with 3 ideal points as vertices has area π.

Theorem H.5.4.5. The area of a triangle having two ideal points as vertices and interior angle α is $\pi - \alpha$.

Theorem H.5.4.7. The area of a triangle with interior angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$.



Figures 5.13, 5.14, and 5.16 from Hitchman.

Note. The smaller the area of a triangle, the closer its angle sum is to the Euclidean case of π. Also, we see that the angle sum of a hyperbolic triangle must be less than π.

Note. Recall that the Euclidean differential of arc length satisfies $ds^2 = dx^2 + dy^2$. Introducing x and y axes to the Poincare disk, we find that

$$ds^2 = \frac{dx^2 + dy^2}{(1-(x^2+y^2))^2}.$$

Exercise. Show that the hyperbolic line from $(x, y) = (-1, 0)$ to $(x, y) = (1, 0)$ is infinite in length.

Note. The *curvature* of a surface is determined from the "metric coefficients." In general, if x and y are a coordinate system on a surface, then the differential of arc length is

$$ds^2 = g_{11}dx^2 + g_{12}dx\,dy + g_{21}dy\,dx + g_{22}dy^2.$$

This determines a matrix of metric coefficients:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

For Euclidean geometry, the matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. For the Poincare disk, the matrix is

$$\begin{pmatrix} 1/(1-(x^2+y^2))^2 & 0 \\ 0 & 1/(1-(x^2+y^2))^2 \end{pmatrix}.$$

From these coefficients, one can show the curvature of the Poincare disk is -4.