IV.2. Power Series Representations of Analytic Functions

Recall. We still have not shown that an analytic function (i.e., a continuously differentiable function) has a power series representation. We do so in this section. Surprisingly, we use integrals to show this.

Proposition IV.2.1. Let \( \varphi : [a, b] \times [c, d] \to \mathbb{C} \) be a continuous function and define \( g : [c, d] \to \mathbb{C} \) by \( g(t) = \int_a^b \varphi(s, t) \, ds \). Then \( g \) is continuous. Moreover, if \( \frac{\partial \varphi}{\partial t} \) exists and is a continuous function on \([a, b] \times [c, d]\) then \( g \) is continuously differentiable and

\[
g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds.
\]

Note. Proposition 2.1 is Leibniz’s Rule from Advanced Calculus. The proof is based in part on the Fundamental Theorem of Calculus.

Lemma IV.2.A. If \( |z| < 1 \) then \( \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} \, ds = 2\pi \).

Note. The text calls the following result “transitory.” However, it will lead us from continuously differentiable to power series.
Proposition IV.2.6. Let $f : G \to \mathbb{C}$ be analytic and suppose $B(a; r) \subseteq G$ ($r > 0$). If $\gamma(t) = a + re^{it}$, and $0 \leq t \leq 2\pi$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw$$

for $|z - a| < r$.

Note. We use Proposition IV.2.6 to introduce series as follows:

$$\frac{1}{w - z} = \frac{1}{w - a} \frac{1}{1 - \frac{z - a}{w - a}} = \frac{1}{w - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{w - a} \right)^n$$

since $|z - a| < r = |w - a|$. We then get

$$\frac{f(w)}{w - z} = \frac{f(w)}{w - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{w - a} \right)^n$$

and with $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$ we have by Proposition IV.2.6 that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \sum_{n=0}^{\infty} \left( \frac{z - a}{w - a} \right)^n \right) \, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \lim_{N \to \infty} \sum_{n=0}^{N} \left( \frac{z - a}{w - a} \right)^n \right) \, dw = \lim_{N \to \infty} \left( \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \sum_{n=0}^{N} \left( \frac{z - a}{w - a} \right)^n \right) \, dw \right)$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \left[ \frac{1}{2\pi i} \int_{\gamma} \left( \frac{f(w)}{w - a} \left( \frac{z - a}{w - a} \right)^n \right) \, dw \right] = \sum_{n=0}^{\infty} \left[ \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \, dw \right) (z - a)^n \right],$$

provided that the equality in red holds. This requires the following.

Lemma IV.2.7. Let $\gamma$ be a rectifiable curve in $\mathbb{C}$ and suppose that $F_n$ and $F$ are continuous on $\{\gamma\}$. If $F$ is the uniform limit of $F_n$ on $\{\gamma\}$ then

$$\int_{\gamma} F = \int_{\gamma} (\lim F_n) = \lim \left( \int_{\gamma} F_n \right).$$
Note. Now to FINALLY resolve our definition of analytic with that used by the real analysts.

**Theorem IV.2.8.** Let $f$ be analytic in $B(a; R)$. Then $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ for $|z - a| < R$ where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence $\geq R$.

Note. Three quick corollaries are:

**Corollary IV.2.11.** If $f : G \to \mathbb{C}$ is analytic and $a \in G$ then $f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n$ for $|z - a| < R$ where $R = d(a, \partial G)$.

**Corollary IV.2.12.** If $f : G \to \mathbb{C}$ is analytic then $f$ is infinitely differentiable.

**Corollary IV.2.13.** If $f : G \to \mathbb{C}$ is analytic and $\overline{B}(a; r) \subseteq G$ then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} dw$$

where $\gamma(t) = a + re^{it}$ and $t \in [0, 2\pi]$.

Note. We can use Corollary 2.13 to evaluate integrals using derivatives.

**Example. (Page 74 #7b.)** Evaluate $\int_{\gamma} \frac{dz}{z - a}$. Let $f(w) = 1$ and $n = 0$. Answer: $2\pi i$. 
Note. The following is an integral step in our proof of the Fundamental Theorem of Algebra.

**Theorem IV.2.14. Cauchy’s Estimate.** Let $f$ be analytic in $B(a; R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a; R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

**Proposition IV.2.15.** Let $f$ be analytic in $B(a; R)$ and suppose $\gamma$ is a closed rectifiable curve in $B(a; R)$. Then $f$ has a primitive in $B(a; R)$ and so $\int_\gamma f = 0.$

*Revised: 10/6/2019*