IV.5. Cauchy's Theorem and Integral Formula

Note. Recall that Proposition IV.2.15 states:

> "Let f be analytic in the disk $B(a;R)$ and suppose that γ is a closed rectifiable curve in $B(a;R)$. Then f has a primitive on $B(a;R)$ and $\int_\gamma f = 0$."

In this section and the next, we look for a converse to this result and we relate this result to winding numbers.

Lemma IV.5.1. Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$, let $F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\,dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Theorem IV.5.4. Cauchy's Integral Formula (First Version).
Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \{\gamma\}$

$$n(\gamma; a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz.$$

Note. For $f(z) \equiv 1$, this follows from the definition of winding number. There is still the surprising fact that the value of the integral only depends on $n(\gamma; a)$.

Note. The following is slightly more general than the Cauchy Integral Formula—First Version and involves more than one rectifiable curve.

Theorem IV.5.6. Cauchy's Integral Formula (Second Version).

Let G be an open subset of the plane and $f : G \to \mathbb{C}$ an analytic function. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \cup\{\gamma_k\}$

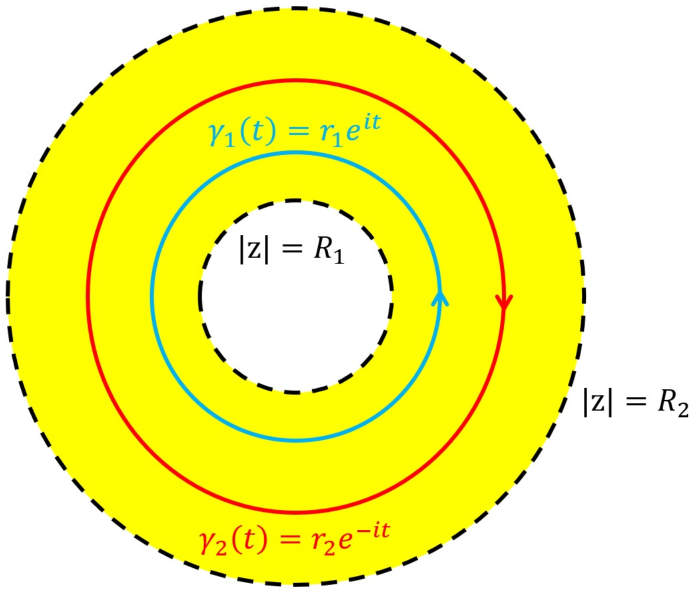
$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \left(\int_{\gamma_k} \frac{f(z)}{z-a}\, dz \right).$$

Note. Notice that Cauchy's Integral Formula deals with integrals of the form $\displaystyle\int_\gamma \frac{f(z)}{z-a}\, dz$. We now deal with versions of "Cauchy's Theorem" which deal with integrals of the form $\displaystyle\int f(z)\, dz$. The first version follows from Theorem 5.6 by applying it to $f(z)(z-a)$.

Theorem IV.5.7. Cauchy's Theorem (First Version).

Let G be an open subset of the plane and $f : G \to \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then $\displaystyle\sum_{k=1}^{m} \left(\int_{\gamma_k} f \right) = 0$.

Note IV-5-A. To illustrate Cauchy's Theorem (First Version), consider region $G\{z \mid R_1 < |z| < R_2\}$ (in yellow in the figure below). Define the curves $\gamma_1(t) = r_1 e^{it}$ and $\gamma_2(t) = r_2 e^{-it}$ where $t \in [0, 2\pi]$ and $R_1 < r_1 < r_2 < R_2$. If $|w| \le R_1$ then $n(\gamma_1; w) = -n(\gamma_2; w) = 1$. If $|w| \ge R_2$ then $n(\gamma_1; w) = n(\gamma_2; w) = 0$. So for all $w \in \mathbb{C} \setminus G$ we have $n(\gamma_1; w) + n(\gamma_2; w) = 0$. For any f analytic on G we then have by Cauchy's Theorem (First Version) that $\int_{\gamma_1} f + \int_{\gamma_2} f = 0$.



Theorem IV.5.8. Let G be an open set and $f : G \to \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_m; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$ and $k \geq 1$,

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \left(\frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\, dz \right).$$

Corollary IV.5.9. (Theorem 5.8 with one curve.) Let G be an open set and $f : G \to \mathbb{C}$ analytic. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$ then for $a \in G \setminus \{\gamma\}$

$$f^{(k)}(a) n(\gamma; a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

Note. Conway says (page 86): "Cauchy's Theorem and Integral Formula is the basic result of complex analysis." Let's use it.

Exercise IV.5.5. Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$, $\int_\gamma (z-a)^{-n}\, dz = 0$.

Example. Compute $\displaystyle\int_{|z|=1} e^z z^{-n}\, dz$. (This is from page 123 of Lars Ahlfors *Complex Analysis*, 3rd Edition, McGraw-Hill Education, 1979).

Note. The text describes the following as a converse of Cauchy's Theorem (Version 1). It is in the sense that it shows that <u>if</u> $\int_\gamma f = 0$ over all γ of a certain form, <u>then</u> f is analytic.

Theorem IV.5.10. Morera's Theorem.
Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every closed triangular path T in G (i.e., T is a closed polygon with 3 sides); then f is analytic in G.

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