VII.8. The Riemann Zeta Function.

Note. In this section, we define the Riemann zeta function and discuss its history. We relate this meromorphic function with a simple pole at $z = 1$ (see Theorem VII.8.14) to, of all things, prime numbers.

Note. Let $z \in \mathbb{C}$ and let $n \in \mathbb{N}$. Then

$$|n^z| = |\exp(z \log n)| = \exp(\text{Re}(z) \log n) = n^{\text{Re}(z)}.$$

So

$$\sum_{k=1}^{n} |k^{-z}| = \sum_{k=1}^{n} \exp(-\text{Re}(z) \log k) = \sum_{k=1}^{n} k^{-\text{Re}(z)}.$$

So if $\text{Re}(z) \geq 1 + \varepsilon$ then

$$\sum_{k=1}^{n} |k^{-z}| \leq \sum_{k=1}^{n} k^{-(1+\varepsilon)} = \sum_{k=1}^{n} \frac{1}{k^{1+\varepsilon}}$$

and the series $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly and absolutely on $G = \{z \mid \text{Re}(z) \geq 1 + \varepsilon\}$ (absolutely by comparing to the p-series with $p = 1 + \varepsilon > 1$ and uniformly by the Weierstrass M-Test [Theorem II.6.2] with $u_n(z) = n^{-z}$ for $\text{Re}(z) \geq 1 + \varepsilon$ and $M_n = 1/n^{1+\varepsilon}$). Now each u_n is continuous on G and so each sum of u_n's is continuous. Since the convergence is uniform, the limit $\sum_{n=1}^{\infty} n^{-z}$ is continuous on G. Also, each u_n is analytic on G, so by Theorem VII.2.1, $\sum_{n=1}^{\infty} u_n(z)$ is analytic on G.

Definition VII.8.1. The Riemann zeta function on $\{z \mid \text{Re}(z) > 1\}$ is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

Note. We will eventually extend $\zeta(z)$ to a function analytic in whole complex plane except for $z = 1$ where it will have a simple pole. The extension is not accomplished by "analytic continuation" (see Chapter IX), but by relating the zeta function to the gamma function.

Note. In 1859, Georg Bernhard Riemann published an 8 page paper, "On the Number of Primes Less Than a Given Magnitude" (Ueber die Anqahl der Primzahlen unter einer gegenen Grösse). A translation of this paper can be found in the appendix of H. M. Edwards' *Riemann's Zeta Function*, Academic Press, Inc. (1974) (this book has now been reprinted by Dover Publications). In the paper, Riemann comments that it is very likely that the complex zeros of the zeta function all have real part equal to 1/2, but that he has been unable to prove this [Edwards, page 6]. This conjecture which concerns the distribution of prime numbers is now the holy grail of open math problems and is known as the Riemann Hypothesis.

Georg Friedrich Bernhard Riemann, 1826–1866

Image from the MacTutor History of Mathematics Archive page on Riemann.

Note. We now relate $\zeta(z)$ to $\Gamma(z)$. Recall that, by Theorem VII.7.15, for $\mathrm{Re}(z) > 0$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-a}\, dt.$$

By replacing the real variable t with nu where $n > 0$ ($t = nu$ and $dt = n\, du$) we get

$$\begin{aligned} \Gamma(z) &= \int_0^\infty e^{-nu}(nu)^{z-1} n\, du \\ &= \int_0^\infty e^{-nu} u^{z-1} n^z\, du \\ &= n^z \int_0^\infty e^{-nt} t^{z-1}\, dt \text{ (in terms of } t) \end{aligned}$$

or

$$n^{-z}\Gamma(z) = \int_0^\infty e^{-nt} t^{z-1}\, dt \text{ for } \mathrm{Re}(z) > 0.$$

If $\mathrm{Re}(z) > 1$ and we sum over $n \in \mathbb{N}$, we get

$$\zeta(z)\Gamma(z) = \left(\sum_{n=1}^\infty n^{-z}\right)\Gamma(z) = \sum_{n=1}^\infty \left(n^{-z}\Gamma(z)\right) = \sum_{n=1}^\infty \left(\int_0^\infty e^{-nt} t^{z-1}\, dt\right). \tag{8.2}$$

We want to take the sum inside the integral and simplify. We need some preliminary results first.

Lemma VII.8.3.

(a) Let $S = \{z \mid \mathrm{Re}(z) \geq a\}$ where $a > 1$. If $\varepsilon > 0$ then there is a number δ, $0 < \delta < 1$, such that for all $z \in S$ we have

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\, dt \right| < \varepsilon$$

whenever $\delta > \beta > \alpha > 0$.

(b) Let $S = \{z \mid \mathrm{Re}(z) \leq A\}$ where $A \in \mathbb{R}$. If $\varepsilon > 0$ then there is a number $\kappa > 1$ such that for all $z \in S$ we have

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1}\, dt \right| < \varepsilon$$

whenever $\beta > \alpha > \kappa$.

Corollary VII.8.4.

(a) If $S = \{z \mid a \leq \mathrm{Re}(z) \leq A\}$ where $1 < a < A < \infty$ then the integral

$$\int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$$

converges uniformly on S.

(b) If $S = \{z \mid \mathrm{Re}(z) \leq A\}$ where $-\infty < A < \infty$, then the integral

$$\int_1^\infty (e^t - 1)^{-1} t^{z-1}\, dt$$

converges uniformly on S.

Note. We now use the uniform convergence of Corollary 8.4 in Equation (8.2).

Note. The following definition and result are from the exercises in Section VII.2.

Definition. Let G be a region, let $a \in \mathbb{R}$ and suppose that $f : [1, \infty] \times G \to \mathbb{C}$ is a continuous function. The integral $F(z) = \displaystyle\int_a^\infty f(t, z)\, dt$ is *uniformly convergent on compact subsets* on G if $\lim_{b\to\infty} \int_a^b f(t, z)\, dt$ exists uniformly for z in any compact subset of G.

Theorem VII.8.A/Exercise VII.2.2. Let G be a region, $a \in \mathbb{R}$, $f : [a, \infty) \times G \to \mathbb{C}$ is a continuous function, and suppose the integral $F(z) = \int_a^\infty f(t, z)\, dt$ is uniformly convergent on compact subsets of G. Suppose for each $t' \in (a, \infty)$, $f(t', z)$ is analytic on G. Then F is analytic on G and

$$F^{(k)}(z) = \int_a^\infty \frac{\partial^k f(t, z)}{\partial z^k}\, dt.$$

Corollary VII.8.B. Let G be a region, $a \in \mathbb{R}$, $f : (0, 1] \times G \to \mathbb{C}$ is a continuous function, and suppose the integral $F(z) = \int_0^1 f(t, z)\, dt$ is uniformly convergent on compact subsets of G. Suppose for each $t' \in (0, 1]$, $f(t', z)$ is analytic on G. Then F is analytic on G and

$$F^{(k)}(z) = \int_a^\infty \frac{\partial^k f(t, z)}{\partial z^k}\, dt.$$

Note. If K is a compact subset of $G = \{z \mid \mathrm{Re}(z) > 1\}$ then K is closed and bounded (by the Heine-Borel Theorem) and so $K \subset \{z \mid a \leq \mathrm{Re}(z) \leq A\}$ for some $a, A \in \mathbb{R}$ with $1 < a \leq A$. With $f(t, z) = (e^t - 1)^{-1} t^{z-1} = (e^t - 1)^{-1} e^{(z-1)\log t}$, we have that for each $t' \in (a, \infty)$, $f(t', z) = (e^t - 1)^{-1}(t')^{z-1}$ is analytic on G. For compact set $K \subset G$ where $K \subset \{z \mid a \leq \mathrm{Re}(z) \leq A\}$ where $1 < a < A < \infty$, so by Corollary VII.8.4(a), $F(z) = \int_0^\infty f(t, z)\, dt = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$ converges uniformly on K and so $F(z)$ is uniformly convergent on compact subsets of G. Therefore, by Theorem VII.8.A/Exercise VII.2.2, $f(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt$ is analytic on G.

Proposition VII.8.5. For $\mathrm{Re}(z) > 1$

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt.$$

Note. We now extend $\zeta(z)$ from $\{z \mid \mathrm{Re}(z) > 1\}$ to all of $\mathbb{C}$ (except $z = 1$). We use the relationship between $\zeta(z)$ and $\Gamma(z)$ given in Proposition VII.8.5 to make this extension.

Note. We now explore the Laurent series for $(e^z - 1)^{-1}$. Notice that

$$\lim_{z\to 0} z(e^z - 1)^{-1} = \lim_{z\to 0} \frac{z}{e^z - 1} = \lim_{z\to 0} \frac{1}{e^z} = 1$$

(L'Hôpital's Rule holds here), so $(e^z - 1)^{-1}$ has a pole of order 1 at $z = 0$ (see Definition V.1.6). By Corollary V.1.18(b), the coefficients a_{-n} in the Laurent series are 0 for $-n \le -2$. By Proposition V.2.4, $a_{-1} = \mathrm{Res}((e^z - 1)^{-1}; 0) = z(e^z - 1)^{-1}\Big|_{z=0} = 1$ (by the limit argument above). So $z(e^z - 1)^{-1}$ (with the removable singularity at $z = 0$ removed) is entire. The coefficient of z in the power series representation of $z(e^z - 1)^{-1}$ is

$$[z(e^z - 1)^{-1}]'\Big|_{z=0} = \frac{(e^z - 1) - ze^z}{(e^z - 1)^2}\Big|_{z=0} = \lim_{z\to 0} \frac{e^z - e^z - ze^z}{2(e^z - 1)z^z}$$
$$= \lim_{z\to 0} \frac{-z}{2(e^z - 1)} = \lim_{z\to 0} \frac{-1}{2e^z} = -\frac{1}{2}.$$

So in the Laurent series for $(e^z - 1)^{-1}$, the constant term is $-1/2$. Therefore, the Laurent series of $(e^z - 1)^{-1}$ is of the form

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^\infty a_n z^n$$

for some $a_1, a_2, \ldots$. So $\dfrac{1}{e^t - 1} - \dfrac{1}{t}$ has a limit as $t \to 0$ (its $-1/2$) and $\dfrac{1}{e^t - 1} - \dfrac{1}{t}$ remains bounded in a neighborhood of $t = 0$. But this implies that the integral

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$$

converges.

Note. We now extend the definition of ζ from $\{z \mid \text{Re}(z) > 1\}$ to $\mathbb{C} \setminus \{1\}$. We do so stepwise.

STEP 1. Extend the definition of ζ from $\{z \mid \text{Re}(z) > 1\}$ to $\{z \mid \text{Re}(z) > 0\}$.

STEP 2. Extend the definition of ζ from $\{z \mid \text{Re}(z) > 0\}$ to $\{z \mid \text{Re}(z) > -1\}$.

STEP 3. Extend the definition of ζ from $\{z \mid \text{Re}(z) > -1\}$ to all of $\mathbb{C} \setminus \{1\}$.

This will give ζ as a meromorphic function with a simple pole at $z = 1$ (see Theorem VII.8.14). At each step, we extend ζ to a new set which overlaps with a previous set; we must confirm that the resulting ζ is well-defined. That is, we must confirm that the definitions are consistent on the overlaps of the sets.

Note. We will explore analytic continuation in Chapter IX. One could use the definition of ζ on $\{z \mid \text{Re}(z) > 1\}$ as $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ and analytic continuation to extend it to $\mathbb{C} \setminus \{1\}$. However, we follow the steps given above. In fact, this is the approach taken by Riemann himself. The following quote is from H. M. Edwards' *Riemann's Zeta Function*, Academic Press (1974), page 9:

> It is interesting to note that Riemann does not speak of the "analytic continuation" of the function $\sum n^{-s}$ beyond the half plane Re $s > 1$, but speaks rather of finding a formula for it which "remains valid for all s." This indicates that he viewed the problem in terms more analogous to the extension [by formulas]. . . than to a piece-by-piece extension of the function in the manner that analytic continuation is customarily taught today. The view of analytic continuation in terms of chains of disks and power series convergent in each disk descends from Weierstrass and is quite antithetical to Riemann's basic philosophy that analytic functions should be dealt with *globally*, not locally in terms of power series.

Lemma VII.8.C. The function $\displaystyle\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > 0\}$.

Note. For $\mathrm{Re}(z) > 0$ we have

$$\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt + (z-1)^{-1} + \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

$$= \int_0^1 \frac{t^{z-1}}{e^t-1}\,dt - \int_0^1 t^{z-2}\,dt + (z-1)^{-1} - \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

$$= \int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt - \frac{1}{z-1} t^{z-1}\Big|_{t=0}^{t=1} + (z-1)^{-1}$$

$$= \int_0^\infty \frac{t^{z+1}}{e^t-1}\,dt - \frac{1}{z-1} + (z-1)^{-1} = \int_0^\infty \frac{t^{z+1}}{e^t-1}\,dt.$$

So the first quantity equals $\zeta(z)\Gamma(z)$ for $\mathrm{Re}(z) > 1$ by Proposition VII.8.5. This motivates us to extend the definition of ζ from $\mathrm{Re}(z) > 1$ to $\mathrm{Re}(z) > 0$, thus completing STEP 1 in our extension of ζ.

Definition. For $\mathrm{Re}(z) > 0$ define the *Riemann zeta function* as

$$\zeta(z) = \frac{1}{\Gamma(z)}\left(\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt + (z-1)^{-1} + \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt\right). \quad (8.7)$$

Note. On $\mathrm{Re}(z) > 0$, ζ is a meromorphic function with a simple pole at $z = 1$. We know that $\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $\mathrm{Re}(z) > 0$ as argued above and $\int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$ is uniformly convergent on compact subsets if $\mathrm{Re}(z) > 0$ by Corollary VII.8.4(b) and so is analytic by Theorem VII.8.A/Exercise VII.2.2. By Proposition V.2.4,

$$\mathrm{Res}(\zeta;1) = (z-1)\zeta(z)\Big|_{z=1} = \lim_{z\to 1}(z-1)\zeta(z) = 1$$

(since the analytic functions given by the integrals produce 0 in the limit and only $\lim_{z\to 1}\frac{z-1}{z-1} = 1$ remains; recall that $\Gamma(1) = 1$). That is, the residue of ζ at $z = 1$ is 1.

Note. We now give an alternative representation of ζ for $0 < \mathrm{Re}(z) < 1$. For $0 < \mathrm{Re} < 1$ then

$$-\int_1^\infty t^{z-2}\,dt - -\frac{t^{z-1}}{z-1}\Big|_{t=1}^{t=\infty} = \frac{1}{z-1}.$$

So by (8.7)

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt - \int_1^\infty t^{z-2}\,dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}\,dt$$

$$= \int_0^\infty \frac{t^{z-1}}{e^t - 1}\,dt \text{ for } 0 < \mathrm{Re}(z) < 1. \qquad (8.8)$$

Lemma VII.8.D. The function $\displaystyle\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) > -1\}$.

Lemma VII.8.E. The function $\displaystyle\int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt$ is an analytic function on $G = \{z \mid \mathrm{Re}(z) < 1\}$.

Note. From (8.8) we have for $0 < \mathrm{Re}(z) < 1$ that

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt \\
&\qquad = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt + \frac{1}{2}\int_0^1 t^{z-1}\,dt - \frac{1}{2z} \\
&\qquad = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt + \frac{1}{2z} t^2\Big|_{t=0}^{t=1} - \frac{1}{2z} \\
&\qquad = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt = \zeta(z)\Gamma(z).
\end{aligned}$$

However, by Lemmas VII.8.D and VII.8.E, the integrals in the first part of this equation are also valid on $\{-1 < \mathrm{Re}(z) < 1\}$. This motivates our next definition which extends ζ to $\{z \mid \mathrm{Re}(z) > -1\}$ and accomplishes STEP 2 of the extension of ζ.

Definition. For $-1 < \mathrm{Re}(z) < 1$ define the *Riemann zeta function* as

$$\zeta(z) = \frac{1}{\Gamma(z)}\left(\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\,dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\,dt\right). \quad (8.9)$$

Note. The term $1/(2z)$ in equation (8.9) makes it appear that ζ may have a pole at $z = 0$. Now each of the integrals in (8.9) represents analytic functions throughout $G = \{z \mid -1 < \text{Re}(z) < 1\}$. Also, $1/\Gamma(z)$ is analytic on G. Notice that $1/(2z\Gamma(z)) = 1/(2\Gamma(z+1))$ by Theorem VII.7.7 (the Functional Equation for Γ), so $1/(2z\Gamma(z))$ is analytic at $z = 0$; its value is $1/(2\Gamma(1)) = 1/2$. So ζ is analytic throughout $\{z \mid -1 < \text{Re}(z) < 1\}$.

Theorem VII.8.13. Riemann's Functional Equation.
For $-1 < \text{Re}(z) < 0$ we have

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin \pi z/2.$$

Note. Conway claims that "The same type of reasoning gives that (8.13) holds for $-1 < \text{Re}(z) < 1$" (page 192). Since the right hand side of Riemann's Functional Equation is analytic in all of $\text{Re}(z) < 0$, we use this as STEP 3 in the extension of ζ to all of $\mathbb{C} \setminus \{1\}$.

Definition. For $\text{Re}(z) < 0$ define the *Riemann zeta function* as

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right).$$

Note. Notice that, by definition, ζ satisfies Riemann's Functional Equation for $\text{Re}(z) < 0$ by definition and satisfies it for $-1 < \text{Re}(z) < 1$ by Conway's claim mentioned above. Treating both sides of Riemann's Functional Equation as analytic functions on $\mathbb{C}\setminus\{1\}$ and the fact that both sides are equal on a subset of $\mathbb{C}\setminus\{1\}$ with a limit point, then by Corollary IV.3.8, the functional equation holds throughout $\mathbb{C}\setminus\{1\}$. We summarize our knowledge of ζ as follows.

Theorem VII.8.14. The zeta function is meromorphic in $\mathbb{C}$ with only a simple pole at $z = 1$ and $\text{Res}(\zeta; 1) = 1$. For $z \neq 1$, ζ satisfies Riemann's Functional Equation.

Note. The gamma function $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$ and is never 0. So $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \ldots$. Now $\zeta(z)$ is analytic at $z = 2, 3, 4, \ldots$, so from Riemann's Functional Equation,

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right),$$

we have for $z = 2, 4, 6, \ldots$ that $\zeta(1-z)\sin(\pi z/2) = 0$ and the simple pole of $\Gamma(1-z)$ cancels with this zero for $z = 2, 4, 6, \ldots$ (otherwise $\zeta(z)$ would not be analytic at $z = 2, 4, 6, \ldots$). So $\zeta(z) \neq 0$ for $z = 2, 4, 6, \ldots$ (since the other factors of $\zeta(z)$ on the right-hand side of the functional equation are nonzero for these values of z). Now $\sin(\pi z/2) = 0$ for $z = -2, -4, -6, \ldots$ and $\Gamma(1-z)$ has no pole at these points, so $\zeta(z) = 0$ for $z = -2, -4, -6, \ldots$. The points $z = -2, -4, -6, \ldots$ are the *trivial zeros* of $\zeta(z)$. By the way, $\zeta(0) = -1/2$ so this covers all even integer values of z (where $\sin(\pi z/2)$ is 0).

Note. Conway comments that: "Similar reasoning gives that ζ has no other zeros outside the closed strip $\{z \mid 0 \leq \text{Re}(z) \leq 1\}$" (page 193).

Definition VII.8.16. The points $z = -2, -4, -6, \ldots$ are the *trivial zeros of* ζ and the strip $\{z \mid 0 \le \text{Re}(z) \le 1\}$ is the *critical strip*.

Note. We are now motivated and have the background to state the Riemann Hypothesis.

The Riemann Hypothesis. If z is a zero of the zeta function in the critical strip then $\text{Re}(z) = 1/2$.

Theorem VII.8.17. Euler's Theorem.
If $\text{Re}(z) > 1$ then

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$

where $\{p_n\}$ is the sequence of prime numbers.

Note. The exercises in the section give several number theoretic properties of $\zeta(z)$. In particular, Exercises VII.8.2–VII.8.5.

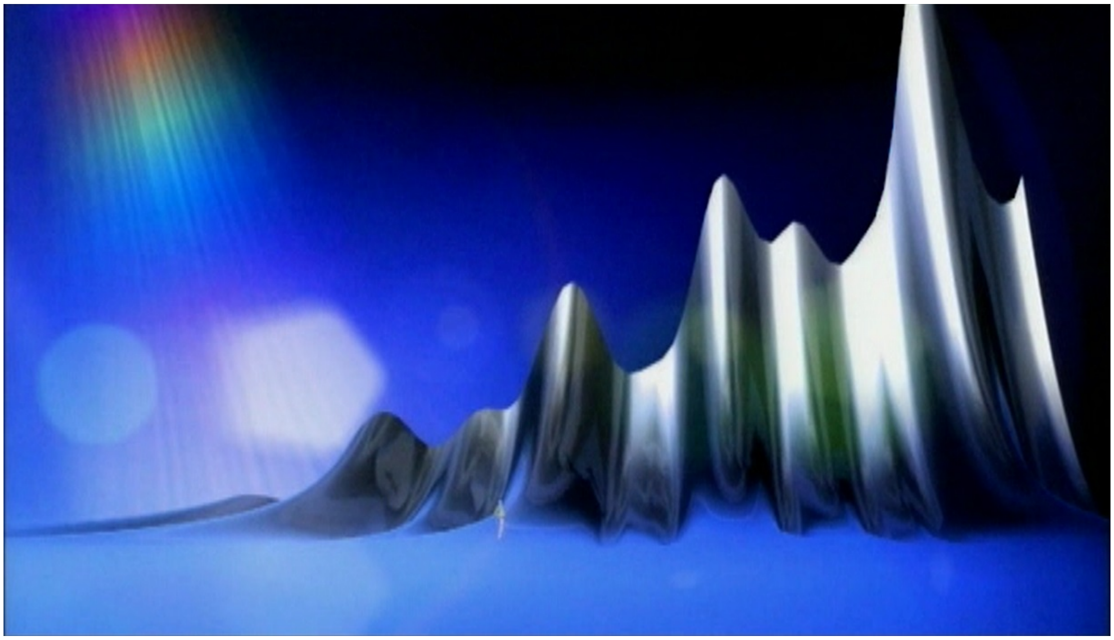
Exercise VII.8.2. Use Euler's Theorem to prove that $\sum_{n=1}^{\infty} p_n^{-1} = \infty$. Notice that this implies that there are an infinite number of primes.

Exercise VII.8.3. Prove that $\zeta^2(z) = \sum_{n=1}^{\infty} d(n)/n^z$ for $\text{Re}(z) > 1$, where $d(n)$ is the number of divisors of n.

Exercise VII.8.4. Prove that $\zeta(z)\zeta(z-1) = \sum_{n=1}^{\infty} \sigma(n)/n^z$ for $\text{Re}(z) > 1$, where $\sigma(n)$ is the sum of the divisors of n.

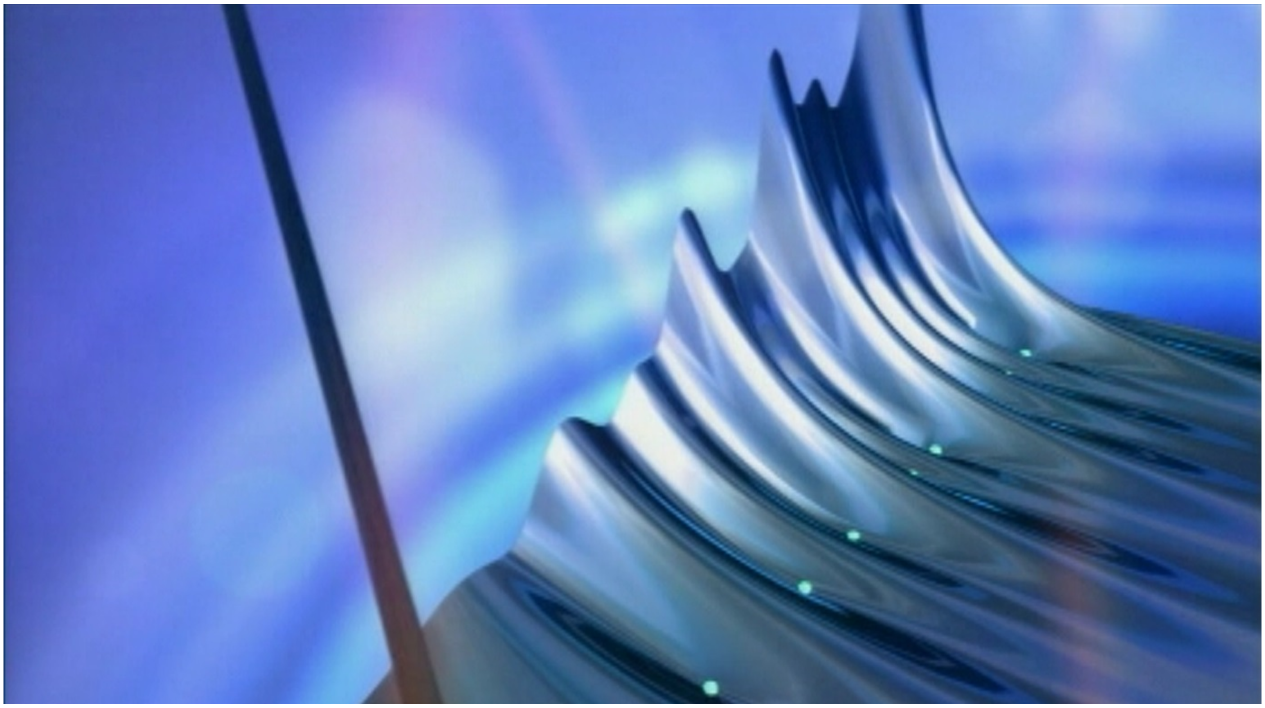
Exercise VII.8.5. Prove that $\zeta(z-1)/\zeta(z) = \sum_{n=1}^{\infty} \varphi(n)/n^z$ for $\text{Re}(z) > 1$, where $\varphi(n)$ is the number of integers less than n and which are relatively prime to n.

Note. In the 2005 BBC and the Open University presented the three part documentary series *The Music of the Primes*, written by Marcus de Sautoy. In it, the modular surface of the Riemann zeta function is given.



The modular surface as viewed in the quadrant where both real and imaginary parts of z are positive.

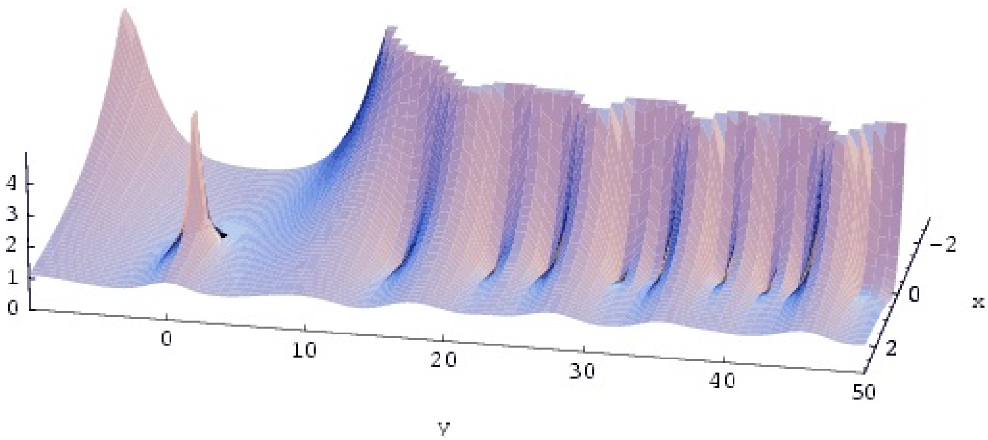
The nontrivial zeros of ζ indicated by white circles.



A view out the line $\mathrm{Re}(z) = 1/2$ with the pole at $z = 1$ visible on the left.

The nontrivial zeros indicated as beams lining up along the line $\mathrm{Re}(z) = 1/2$.

Note. Another view of the modular surface is given at the Michigan State University website (accessed 4/8/2022):



In this view of the modular surface $|\zeta(x+iy)|$, the pole is on the left and the trivial zeros are off in the valley on the left behind the pole. Several of the nontrivial zeros are visible running off to the right. The nontrivial zeros of ζ with imaginary parts between 0 and 50 are $1/2 + iy$ where (to two decimal places) y is: 14.13, 21.02, 25.01, 30.42, 32.94, 37.59, 40.92, 43.33, 48.01, and 49.77 (from "The First 100 (non trivial) Zeros of the Riemann Zeta Function" webpage; accessed 4/8/2022).

Note. The main purpose of Riemann's 1859 paper was to find an asymptotic approximation of $\pi(x)$, the number of prime numbers less than or equal to x. Two functions which approximate $\pi(x)$ are $x/\log x$ and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$. We denote this "approximation" as $\pi(x) \sim x/\log x$ and $\pi(x) \sim \mathrm{Li}(x)$. We have $x \log x \sim \mathrm{Li}(x)$, so that the claims $\pi(x) \sim x/\log x$ and $\pi(x) \sim \mathrm{Li}(x)$ are equivalent (since $\sim$ is an equivalence relation). This approximation problem is called The Prime Number Theorem, and was proved independently by Jacques Hadamard and Charles de la Vallée Poussin in 1896. For more details, see my supplemental online notes for use in Elementary Number Theory (MATH 3120) and Number Theory (MATH 5070) on Supplement. The Prime Number Theorem—History. More mathematical details are given in my online supplemental notes on The Prime Number Theorem (these notes are in preparation, as of spring 2022). The interest in the Riemann Hypothesis stems from the fact that the *quality* of the approximation (that is, the size of the error term in the approximation) is related to the nontrivial zeros of the zeta function. Knowing the precise nontrivial zeros of the zeta function yields a precise formula for $\zeta(z)$. For more details, so another of my online supplements to the number theory classes on Supplement. The Riemann Hypothesis—History. Another useful and readable source of information on the Prime Number Theorem and the Riemann Hypothesis is John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003); my online supplemental notes heavily quote this source.