XI.2. The Genus and Order of an Entire Function.

Note. From the Weierstrass Factorization Theorem, we know that an entire function f with nonzero zeros $\{a_1, a_2, \ldots\}$ can be factored as $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}(z/z_n)$ where g is entire (so all zeros of f are in the z^m and $\prod E_{p_n}(z/a_n)$ terms). In this section we put restrictions on the "rate of growth" of f and consider the resulting class of functions.

Definition XI.2.1. Let f be an entire function with zeros $\{a_1, a_2, \ldots\}$ repeated according to multiplicity and arranged such that $|a_1| \le |a_2| \le \cdots$. Then f is of *finite rank* if there is $p \in \mathbb{Z}$ such that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$. If p is the least integer such that this occurs, then f is *of rank* p. A function with only a finite number of zeros has rank 0. A function is of *infinite rank* if it is not of finite rank.

Note. Recall from Theorem VII.5.12 that a sequence $\{a_n\}$ such that $\lim_{n\to\infty} |a_n| = \infty$, where $a_n \neq 0$ for all $n \ge 1$, for which no complex number is repeated an infinite number of times in the sequence for which there is a sequence $\{p_n\}$ of nonnegative integers such that $\sum_{n=1}^{\infty} (r/|z_n|)^{p_n+1} < \infty$ for all $r > 0$, we have that $P(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a)$ is an entire function with zeros only at the points a_n and the multiplicity of a zero is given by the number of times it is repeated in the sequence. Here, $E_p(z) = (1 - z)\exp(z + z^2/2 + \cdots + z^p/p)$ for $p \ge 1$ are the "elementary functions."

Note. Recall that the Weierstrass Factorization Theorem (Theorem VII.5.14) states that if f is an entire function with nonzero zeros $\{a_n\}$ repeated according to multiplicity and a zero of 0 of multiplicity m, then $f(z) = z^m e^{g(z)} P(z)$ where g is an entire function and P is as given above.

Note. If entire function f has finite rank p and nonzero zeros $\{a_n\}$ repeated according to multiplicity, then be definition there is integer p such that

$$\sum_{n=1}^{\infty} |a_n|^{-(p+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{|a_n|}\right)^{p+1} < \infty.$$

For $r > 0$, since $\lim_{n\to\infty} |a_n| = \infty$, then $r/|a_n|$ is "eventually" less than 1 and so $\sum_{n=1}^{\infty} |r/a_n|^{p+1}$ converges (by the Direct Comparison Test as compared to a geometric sequence). So with each $p_n = p$, Theorem VII.5.12 gives $P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$ and the Weierstrass Factorization Theorem (Theorem VII.5.13) gives that $f(z) = z^m e^{g(z)} P(z)$. Now if p_1 is any integer larger than the rank, then we similarly have $f(z) = z^m e^{g_1(z)} P_1(z)$ where $P_1(z) = \prod_{n=1}^{\infty} E_{p_1}(z/a_n)$ so that the factorization of f is not unique.

Definition XI.2.4. Let f be an entire function of rank p with zeros $\{a_1, a_2, \ldots\}$. Then the product $P(z) = \prod_{n=1}^{\infty} E_p(z/a_n)$ is the *standard form for* f.

Note. For f of rank p, the Weierstrass factorization $f(z) = z^m e^{g(z)} P(z)$ where P is standard form for f is unique except that entire function g can be modified by adding integer multiples of $2\pi i$ (but this does not change the factor $e^{g(z)}$).

Definition XI.2.5. An entire function f has *finite genus* if f has finite rank and if $f(z) = z^m e^{g(z)} P(z)$ where P is in standard form for f and the resulting function g is a polynomial. If p is the rank of f and q if the degree of polynomial g, then $\mu = \max\{p, q\}$ is the *genus* of f.

Note. Although function g is not uniquely defines, if it is a polynomial then its degree is uniquely defined, so that the genus of an entire function of finite genus is well-defined.

Note. The following theorem shows that an entire function of genus μ has a certain bound on its "rate of growth."

Theorem XI.2.6. Let f be an entire function of genus μ. For each positive number α there is a number r_0 such that for $|z| > r_0$ we have $|f(z)| < \exp(\alpha |z|^{\mu+1})$.

Note. For an entire function f we define the quantity $M(r) = \max_{|z|=r}\{|f(z)|\}$. Notice that by the Maximum Modulus Theorem, $M(r)$ is a strictly increasing function of r. We can interpret Theorem XI.2.6 as saying that entire function $f(z) = z^m e^{g(z)} P(z)$ for which the rate of growth of the zeros of f is restricted (that is, f is of finite rank) and for which g is a polynomial (that is, f is of finite genus), then $M(r)$ is dominated ("eventually," i.e. for $|z| > r_0$) by $\exp(\alpha r^{\mu+1})$ for all $\alpha > 0$ and for μ the genus of f.

Definition XI.2.13. An entire function f is of *finite order* if there is a positive constant a and an $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. If such a and r_0 do not exist then f is of *infinite order*. If f is of finite order then $\lambda = \inf\{a \mid |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\}$ is the *order* of f.

Note. If f has finite order λ, we will see a formula for computing the order in Proposition XI.2.15 below. In fact, an example of an entire function of order n is $\exp(z^n)$.

Proposition XI.2.14. Let f be an entire function of finite order λ. If $\varepsilon > 0$ then $|f(z)| < \exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large, and z can be found with $|z|$ as large as desired such that $|f(z)| \geq \exp(|z|^{\lambda+\varepsilon})$.

Note. An easy corollary to Theorem XI.2.6 is the following which follows by taking $\alpha = 1$ in Theorem XI.2.6.

Corollary XI.2.16. If f is an entire function of finite genus μ then f is of finite order λ where $\lambda \geq \mu + 1$.

Note. We defined finite genus for functions of finite rank but we defined order for every entire function (though it might be infinite). Corollary XI.2.16 shows that finite genus implies finite order. In the next section we will show that finite order implies finite genus in Hadamard's Factorization Theorem.

Note. The next proposition gives a way to compute the order of an entire function. We leave the proof as Exercise XI.2.A.

Proposition XI.2.15. Let f be an entire function of order λ and let $M(r) = \max_{|z|=r}\{|f(z)|\}$. Then

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

Example. For $f(z) = \exp(\exp z) = e^{e^z}$ we have

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log(\log e^{e^r})}{\log r} = \lim_{r\to\infty} \frac{r}{\log r} = \infty.$$

Similarly, for $n \in \mathbb{N}$, $g(z) = \exp(z^n) = e^{z^n}$ is of order n. More generally, if p is a polynomial of degree n then $\exp(p(z)) = e^{p(z)}$ is of order n.

Note. In Ralph Boas' *Entire Functions*, Academic Press (1954), the formula in Proposition XI.2.15 is given as the definition of the order of an entire function. He went on to give the following definition concerning a "finer" parameter concerning the rate of growth of an entire function of finite order. See page 8 of Boas' book.

Definition. Let f be an entire function of positive finite order λ and let $M(r) = \max_{|z|=r}\{|f(z)|\}$. then f is of *type* τ if

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\lambda}.$$

The collection of all *functions of exponential type* τ, denoted $\mathcal{E}_\tau$, includes all entire functions of order 1 and type less than or equal to τ, as well as all entire functions of order less than 1.

Note. The class of all entire functions of exponential type τ forms a "linear space," as does the class of all polynomials of degree n or less, denoted $\mathcal{P}_n$.

Note. Boas also relates the order and type of $f(z) = \sum_{n=1}^{\infty} a_n z^n$ to the coefficients a_n in the following two results. See pages 9–11 of Boas' book.

Theorem XI.2.A. (Boas' Theorem 2.2.2.)

The entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of finite order λ if and only if

$$\lambda = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}.$$

Theorem XI.2.B. (Boas' Theorem 2.2.10.)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\nu = \limsup_{n\to\infty} n|a_n|^{\lambda/n}$. If $0 < \nu < \infty$ then f is of order λ and type τ if and only if $\nu = e\tau\lambda$.

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