Supplement. Ordering the Complex Numbers

Note. In this supplement, we recall the definition of an ordering of a field. We prove some elementary properties of an ordered field concerning the additive identity 0 and the multiplicative identity 1. We then consider i such that $i^2 = -1$ and show that if such an i is in a field, then the field does not admit an ordering.

Note. The following definition of an ordering of a field $\mathbb{F}$ is from James R. Kirkwood, *An Introduction to Analysis*, 2nd edition (PWS Publishing Company and Waveland Press, 1995). This is the reference I use for Analysis 1 (MATH 4217/5217) and the definition can by found in my online notes for that class on Section 1.2. Properties of the Real Numbers as an Ordered Field.

Definition. A field $\mathbb{F}$ is *ordered* if there is a nonempty set $P \subset \mathbb{F}$ (called the *positive subset*) for which

(1) $a, b \in P$ implies $a + b \in P$

(2) $a, b \in P$ implies $ab \in P$

(3) $a \in \mathbb{F}$ implies exactly one of the following: $a \in P$, $-a \in P$, or $a = 0$.

For $a, b \in \mathbb{F}$, if $b - a \in P$, then we say $a < b$.

Note. Property (1) is closure under addition, Property (2) is closure under multiplication, and Property (3) is called the *Law of Trichotomy*. We now prove some of the properties of an ordered field.

Theorem 1. If $a < b$ and $c > 0$, then $ac < bc$.

Proof. Since $a < b$, then $b - a \in P$ by the definition of $<$. Since $c > 0$, then $c \in P$. By Property 2 (closure of P under multiplication), $(b - a)c = bc - ac \in P$. Therefore, by the definition of $<$, we have $ac < bc$ as claimed. ∎

Theorem 2. $0 < 1$

Proof. ASSUME not. That is, assume $1 \leq 0$. Then $1 < 0$ (every field has at least two distinct elements: the additive identity 0 and the multiplicative identity 1; so $0 \neq 1$). Then $0 - 1 = -1 \in P$. By Property 2 (closure of P under multiplication), $(-1)(-1) \in P$ and $(-1)(-1) = 1$ (a property of fields), so that $1 \in P$. But this is a CONTRADICTION, so our assumption that $1 < 0$ is false and, since $0 \neq 1$, then the assumption that $1 \leq 0$ is false, and hence $0 < 1$, as claimed. ∎

Note. In the next two results, we assume that $\mathbb{C}$ is an ordered field. We denote as i a complex number such that $i^2 = -1$. Under the assumption, by the Law of Trichotomy, either $i \in P$, $-i \in P$, or $i = 0$ (but $0^2 = 0 \neq -1$, so $i \neq 0$).

Corollary 1. Suppose that $\mathbb{C}$ is an ordered field with positive set P, and let $i \in \mathbb{C}$ be such that $i^2 = -1$. Then $i \notin P$.

Proof. ASSUME not. That is, assume that $i \in P$. Then by Property 2 (closure of P under multiplication), $(i)(i) = i^2 = -1 \in P$. But this CONTRADICTS Theorem 2, so our assumption that $i \in P$ is false, and hence $i \notin P$, as claimed. ∎

Corollary 2. Suppose that $\mathbb{C}$ is an ordered field with positive set P, and let $i \in \mathbb{C}$ be such that $i^2 = -1$. Then $-i \notin P$.

Proof. ASSUME not. That is, assume that $-i \in P$. By Property 2 (closure of P under multiplication), $(-i)(-i) = (i)(i) = i^2 = -1 \in P$. But this CONTRADICTS Theorem 2 (which states that $0 < 1$, or $1 - 0 = 1 \in P$), so our assumption that $-i \in P$ is false, and hence $-i \notin P$, as claimed. ∎

Note. In the previous two corollaries combine with the Law of Trichotomy to show that $\mathbb{C}$ is not an ordered field, as follows.

Theorem 3. The complex numbers $\mathbb{C}$ do not form ordered field.

Proof. ASSUME that $\mathbb{C}$ is an ordered field. Of course there is $i \in \mathbb{C}$ such that $i^2 = -1$. Since $0^2 = 0 \neq -1$, then $i \neq 0$. By Corollary 1, $i \notin P$. By Corollary 2, $-i \notin P$. But then $i \in \mathbb{C}$ does not satisfy the Law of Trichotomy, a CONTRADICTION. So the assumption that $\mathbb{C}$ is an ordered field is false. That is, $\mathbb{C}$ is not an ordered field, as claimed. ∎

Note. The above result does not imply that we cannot put *any* sort of ordering on $\mathbb{C}$! Only that (in informal terms) we cannot put a *useful* ordering on $\mathbb{C}$. Consider $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Define $z_1 \prec z_2$ if and only if either (1) $a_1 < a_2$ or (2) $a_1 = a_2$ and $b_1 < b_2$. This is called the *lexicographic* ordering of $\mathbb{C}$. This is because it is similar to the way words are alphabetized. For any $z_1, z_2 \in \mathbb{C}$ we have exactly one of the following: (1) $z_1 \prec z_2$, (2) $z_2 \prec z_1$, or (3) $z_1 = z_2$. Also, if $z_1 \prec z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Note. With the lexicographic ordering, $0 \prec i$ (i.e., i is "positive; if such a thing makes sense). With $a = 0$, $b = i$, and $c = i$, we have: $a \prec b$ and $0 \prec c$, but $ac = 0 \prec bc = (i)(i) = i^2 = -1$ which is not the case! So Theorem 2 does not hold for the lexicographic ordering of $\mathbb{C}$.

Note. The lexicographic ordering is also useless in defining completeness. Consider the set $A = \{z \in \mathbb{C} \mid z \prec 0\}$. The set A has an $\prec$-upper bound (say 1), and has a $\prec$-least upper bound (namely 0). Consider the set $B = \{z = a+ib \mid a \leq 0 \text{ and } b \in \mathbb{R}\}$. Then B has an $\prec$-upper bound (say 1), but B has no $\prec$-least upper bound!

Note. Perhaps you are familiar with the idea of a *well-ordering* from set theory. This is *not* related to the type of ordering we are talking about! An *ordering* of a set A is a binary relation $\preceq$ on the set such that the relation is (1) reflexive ($a \preceq a$ for all $a \in A$), (2) antisymmetric ($a \preceq b$ and $b \preceq a$ imply $a = b$), and (3) transitive ($a \preceq b$ and $b \preceq c$ imply $a \preceq c$). An ordering $\preceq$ of a set A is a *total ordering* if for any $a, b \in A$, either $a \preceq b$ or $b \preceq a$ (that is, any two elements of A are *comparable*). An example of a totally ordered set is the set $A = \mathbb{R}$ with ordering $\leq$.

Note. A total ordering $\preceq$ of a set A is a *well-ordering* if every nonempty subset of A has a $\preceq$-least element. That is, if $B \subset A$ then there exists $m \in B$ such that $m \preceq b$ for all $b \in B$. An example of a well-ordered set is $\mathbb{N}$ with ordering $\leq$. Notice that $\leq$ does not well-order $\mathbb{R}$.

Note. The *Well-Ordering Principle* states that every set can be well-ordered. This result is equivalent to the Axiom of Choice. It is therefore true that $\mathbb{C}$ can be "well-ordered," but this should not be confused with the idea of ordering $\mathbb{C}$ in a way that generalizes the ordering of $\mathbb{R}$. The ideas of ordering and well-orderings of sets (and the Well-ordering Principle) are covered in Introduction to Set Theory (not an official ETSU class) in Section 7.1, "Well-Ordered Sets" (these notes are currently [fall 2023] in preparation).

Note. The lexicographic ordering of $\mathbb{C}$ is an example of a total-ordering. However, as shown above, NO ordering of $\mathbb{C}$ can satisfy the Law of Trichotomy, and hence there is no (useful) way to extend the ordering of $\mathbb{R}$ to an ordering of $\mathbb{C}$.

References

1. K. Hrbacek and T. Jech, *Introduction to Set Theory*, 2nd Edition Revised and Expanded, Monographs and Textbooks in Pure and Applied Mathematics #85, Marcel Dekker, Inc., 1984.

2. J. R. Kirkwood, *An Introduction to Analysis*, 2nd edition, PWS Publishing Company and Waveland Press, Inc., 1995.

3. "Among any two integers or real numbers one is larger, another smaller. But you can't compare two complex numbers," Copyright 1996–2009 Alexander Bogomolny. This is online at Cut-the-Knot.org (accessed 8/31/2023).

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