APPLIED MATH II TEST 1
Spring 1997

NAME ______________________ STUDENT NUMBER ______________________

1. Answer each of the following.

a. State the definition of a *second order PDE* in two variables.

b. State the definition of an *operator* (what is domain and range?) and a *linear operator*.

c. State the definition of a *boundary condition* for a PDE in unknown function u on region R with boundary D. State the definitions of *Dirichlet*, *Neumann*, and *Robin* boundary conditions.

2. Answer each of the following.

a. State the definition of a *well-posed PDE.*

b. State the "Maximum Principle" (and include the setting to which it applies).

c. Why will we be studying Fourier series? That is, how do Fourier series naturally arise in what we have been studying?

3. Prove one of the following two:

a. Prove that the nonhomogeneous Dirichlet problem for the diffusion equation:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \text{ for } 0 \leq x \leq l, t > 0 \\ u(x,0) &= \Phi(x) \\ u(0,t) &= g(t), u(l,t) = h(t) \end{aligned}$$

has at most one solution.

b. Prove that

$$u(x,t) = \frac{1}{2}\Phi(x+ct) + \frac{1}{2}\Phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)\,ds$$

is a solution to

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < +\infty \\ u(x,0) &= \Phi(x), u_t(x,0) = \Psi(x). \end{aligned}$$

4, 5. Do two of the following three:

a. Use seperation of variables $u(x,t) = X(x)T(t)$ to find $T(t)$ in the wave equation.

b. Recall that the general solution to the equation $u_t = ku_{xx}$ is (by seperation of variables)

$$u(x,t) = Ae^{-\lambda kt}(C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)).$$

Solve the mixed boundary value problem:

$$\begin{aligned} u_t &= ku_{xx}, 0 < x < l \\ u(0,t) &= u_x(l,t) = 0 \end{aligned}$$

c. Use the Divergence Theorem to show that a neccesary condition for the Neumann problem

$$\begin{aligned} \nabla^2 u &= f(x,y,z) \text{ in } D \\ \frac{\partial u}{\partial n} &= 0 \text{ on bdy } D \end{aligned}$$

to have a solution is that

$$\iiint_D f(x,y,z)\, dx\, dy\, dz = 0.$$

Recall that the Divergence Theorem says:
Let T be a solid with the boundary of the surface S which consists of finitely many smooth pieces. If the components of $\mathbf{v} = \mathbf{v}(x, y, z)$ are continuously differentiable on T, then

$$\iint_S (\mathbf{v} \cdot \mathbf{n})\, ds = \iiint_T (\nabla \mathbf{v})\, dx\, dy\, dz$$

where $\mathbf{n}$ is a unit vector normal to surface S. (Here we use "∇" to represent the divergence of a vector function, ∇^2 to represent the Laplacian operator [we used Δ to represent this operator in class], and **bold faced** type to denote vectors). Assume the domain D above satisfies the hypotheses of the Divergence Theorem.