Chapter 1. Where PDEs Come From

Section 1.1. What is a PDE?

Note. In this section we give some introductory definitions and give some elementary examples.

Note. If $u : \mathbb{R}^2 \to \mathbb{R}$, say $u = u(x, y)$, the denote

$$\frac{\partial u}{\partial x} = u_x,\ \frac{\partial u}{\partial y} = u_y,\ \frac{\partial^2 u}{\partial x^2} = u_{xx},\ \frac{\partial^2 u}{\partial xy} = u_{yx},\ \frac{\partial^2 u}{\partial y^2} = u_{yy}.$$

Note. Euler's Theorem states:

If $u(x, y)$ and u_x, u_y, u_{xy}, and u_{yx} are defined on an open region containing a point (a, b) and all are continuous at point (a, b) then

$$u_{xy}(a, b) = u_{yx}(a, b).$$

Definition. A *first order PDE* in the dependent variable u and the independent variables x, y is $F(x, y, u, u_x, u_y) = 0$. A *second order PDE* in two variables is $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$. A *solution* to either of these two types of PDEs is a function $u(x, y)$ that satisfies the equation in some region in $\mathbb{R}^2$.

Note. We assume throughout that mixed partials are equal (and, therefore, that the hypotheses of Euler's Theorem are satisfied).

Definition. An *operator* is a function that maps functions to functions. An operator $\mathcal{L}$ is *linear* if for all functions u, v in the domain of $\mathcal{L}$, and for all (real) constants a, b we have:

$$\mathcal{L}[au + bv] = a\mathcal{L}[u] + b\mathcal{L}[v].$$

If $\mathcal{L}$ is a linear operator, then the equation $\mathcal{L}[u] = 0$ is a *homogeneous linear equation* and the equations $\mathcal{L}[u] = g$, where $g \not\equiv 0$ is a function of the independent variable(s), is a *nonhomogeneous linear equation.*

Example/Definition. The operator $\mathcal{L} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is a linear operator since on the collection of twice differentiable functions of two variables,

$$\mathcal{L}[au + bv] = \frac{\partial^2}{\partial x^2}[au + bv] + \frac{\partial^2}{\partial y^2}[au + bv] = a\frac{\partial^2}{\partial x^2}[u] + b\frac{\partial^2}{\partial x^2}[v] + a\frac{\partial^2}{\partial y^2}[a] + b\frac{\partial^2}{\partial y^2}[v]$$

$$= a\left(\frac{\partial^2}{\partial x^2}[u] + \frac{\partial^2}{\partial y^2}[u]\right) + b\left(\frac{\partial^2}{\partial x^2}[v] + \frac{\partial^2}{\partial y^2}[v]\right) = a\mathcal{L}[u] + b\mathcal{L}[v].$$

This linear operator is called the *Laplacian* which arises in heat and potential problems.

Example/Definition. The PDE $\dfrac{\partial^2}{\partial x^2}[u] + \dfrac{\partial^2}{\partial y^2}[u] = 0$ is a homogeneous linear equation called *Laplace's equation* (or the *potential equation*). It describes the distribution of heat at equilibrium. It also describes gravitational and electrostatic potential. A function satisfying Laplace's equation is said to be *harmonic.*

Example. The PDE $u_t + uu_x + y_{xxx} = 0$ (the dispersive wave equation) is nonlinear.

Theorem. If $\mathcal{L}$ is a linear operator and u and v are solutions of $\mathcal{L}[x] = 0$, then any linear combination of u and v is also a solution.

Solution. We are given that $\mathcal{L}[u] = 0$ and $\mathcal{L}[v] = 0$. Let $a, b \in \mathbb{R}$. Then

$$\mathcal{L}[au + bv] = \mathcal{L}[au] + \mathcal{L}[bv] = a\mathcal{L}[u] + b\mathcal{L}[v] = a0 + b0 = 0,$$

so the linear combination $au + bv$ is also a solution of $\mathcal{L}[x] = 0$. ∎

Note/Definition. The above result tells us that any linear combination of solutions of a homogeneous linear PDE is again a solution. This is called the *superposition principle.*

Example. Solve $u_{xx} + u = 0$.

Solution. If this were an ordinary differential equation ("ODE"), then we would have $u(x) = c_1 \cos x + c_2 \sin x$ for arbitrary constants $c_1, c_2 \in \mathbb{R}$. But since here we have u as a function of two variables, $u = u(x, y)$, and the differentiation is with respect to x, then we have the constants here as constant with respect to x so that they are in fact functions of y. So the solution is

$$u(x, y) = f(y) \cos x + g(y) \sin x,$$

where f and g are arbitrary functions of y. □

Note. Whereas a second order linear ODE has 2 arbitrary constants, the previous second order linear PDE has two arbitrary functions.