Section 1.2. First-Order Linear Equations

Note. In this section, we give some examples of solving PDEs. We primarily make use of certain change of variable "tricks."

Example. Sole the PDE $au_x + bu_y = 0$, where not both a and b are zero.

Solution. We make the change of variables $v = ax + by$ and $w = bx - ay$. Then by the Chain Rule:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = au_v + bu + w$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = bu_v - au_w.$$

Then $au_x + bu_y = (a^2 + b^2)u_v = 0$. So $u_v = 0$ and hence $u(v, w) = f(w) = f(bx - ay)$ where f is an arbitrary differentiable function. So $u(x, y) = f(bx - ay)$. □

Example. Solve $\begin{cases} u_t + 3u_x = 0 \\ u(0, x) = 1 - x^2. \end{cases}$

Solution. As above, we get $u(t, x) = f(3t - x)$. With $t = 0$, $f(-x) = 1 - x^2$ which implies $f(x) = 1 - x^2$ and so $f(3t - x) = 1 - (3t - x)^2$ or

$$u(t, x) = 1 - (3t - x)^2 = 1 - 9t^2 + 6tx - x^2.$$

□

Definition. The *gradient* of $f(x, y)$ is the vector function $\nabla f = f_x \,\hat{\imath} + f_u \,\hat{\jmath}$. The *directional derivative* of f in direction $\vec{d}$ (a unit vector) is

$$D_{\vec{d}}[f] = \nabla f \cdot \vec{d} = d_1 f_x + d_2 f_y$$

where $\vec{d} = \langle d_1, d_2 \rangle$.

Example. Page 9 Example 3. Solve $u_x + 2xy^2 u_y = 0$.

Solution. This PDE can be interpreted as "the directional derivative of u in the direction $\vec{d} = \langle 1, 2xy^2 \rangle$ is 0" (notice that since the derivative is set equal to 0, the length of $\vec{d}$ need not be 1). So the PDE is satisfied on curves in the xy-plane with tangent vectors $\langle 1, 2xy^2 \rangle$. This curves satisfy

$$\frac{dy}{dx} = \frac{2xy^2}{1} = 2xy^2.$$

We have $dy/y^2 = 2x\,dx$ (separating variables) or $-1/y = x^2 + c$ or $y = -1/(c + x^2)$ for some constant c. Notice

$$\frac{d}{dx}\left[u\left(x, -\frac{1}{c + x^2}\right)\right] = u_x + u_y \frac{2x}{(c + x^2)^2} = u_x + 2xy^2 u_y = 0,$$

so $u(x, y)$ is constant on curves of the form $y = -1/(c + x^2)$. Notice that

$$u(x, y)|_{x=0} = u\left(x, -\frac{1}{c + x^2}\right)\bigg|_{x=0} = u(0, -1/c).$$

So $u(x, y) = u(1, -1/c)$ on curves $y = -1/(c + x^2)$ and therefore $u(x, y) = f(c)$ for some differentiabe function f and where $c = -x^2 = 1/y$. Therefore

$$u(x, y) = f(x^2 + 1/y),$$

where f is an arbitrary differentiable function. □

Note. The above process is called the "Geometric Method." I reduces a PDE of the form $a(x,y)u_x + b(x,y)u_y = 0$ to the ODE $\dfrac{dy}{dx} = \dfrac{b(x,y)}{a(x,y)}$.

Example. Page 9 Number 7. Solve $au_x + bu_y + cy = 0$.

Solution. We make the change of variables $v = ax + by$ and $w = bx - ay$, and get (as in the first example of this section)

$$au_x + bu_y + cu = (a^2 + b^2)u_v + cu = 0.$$

We can treat this as a first order ODE in v to get $u(v,w) = g(w)e^{-cv/(a^2+b^2)}$. Therefore

$$u(x,y) = g(bx - ay)e^{-c(ax+by)/(a^2+b^2)}$$

where g is an arbitrary differentiable function. Notice the back of the book gives $u(x,y) = e^{-cx/a}f(bx - ay)$ where f is an arbitrary differentiable function.

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