Section 1.3. Flows, Vibrations, and Diffusions

Note. In this section, we derive several examples of PDEs inspired by physical models. We use some well-known physical principles, like Newton's Second Law of Motion, in the derivations.

Note. We first consider the model of "simple transport." Suppose a fluid flows through a pipe of fixed cross section at a constant rate of c (spatial unit/temporal unit). Suppose there is a substance suspended in the fluid with a concentration of $u(x,t)$ (mass/spatial unit) at position x and time t. Notice that we are ignoring the friction of the fluid with the pipe and we also assume no diffusion. The amount of the dissolved substance in the pipe in the segment from $x = a$ to $x = b$ is $\int_a^b u(x,t)\,dx$ and is a function of time. So the amount of substance in the pipe from $x = 0$ to $x = b$ at time t is $\int_0^b u(x,t)\,dx$. At time $t+k$, this amount of the substance has moved to the segment of the pipe from $x = ch$ to $x = b + ch$ and so

$$\int_0^b u(x,t)\,dx = \int_{ch}^{b+ch} u(x,t+h)\,dx.$$

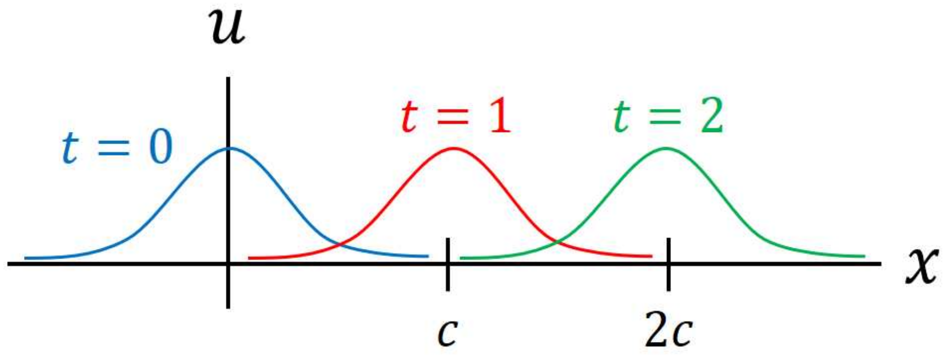
Differentiating with respect to b and applying the Fundamental Theorem of Calculus give $u(b,t) = u(b+ch, t+h)$. Differentiating this with respect to h gives

$$0 = cu_x(b+ch, t+h) + u_t(b+ch, t+h).$$

At $h = 0$ we get $u_t(b,t) + cu_x(b,t) = 0$. Since b is an arbitrary value of x, then $u_t(x,t) + xu_x(x,t) = 0$, or simply

$$\boxed{u_t + cu_x = 0.}$$

Note. As seen in Section 1.2, the solution to the PDE $u_t + cu_x = 0$ is $f(x - ct)$ where$f$ is an arbitrary differentiable function. If we have supplemental conditions (say initial conditions) such as the initial distribution of concentration $u(x, 0)$, then we need $f(x) = u(X, 0)$. Notice that this implies the following behavior.



Note. We next consider vibrating strings and the wave equation. First, we review some physics.

Note. If a wire is pulled by its ends with forces of size F adn $-F$ then the wire attains an internal "force" called *tension*. Notice that under these conditions, the wire does not move since the net force is 0. If we cut the wire at some point and attach a spring scale, then we can measure tension.

Note. Recall that Newton's Second Law of Motion states:

> If a force F is applied to a mass m then the mass experiences an acceleration a where $F = ma$.

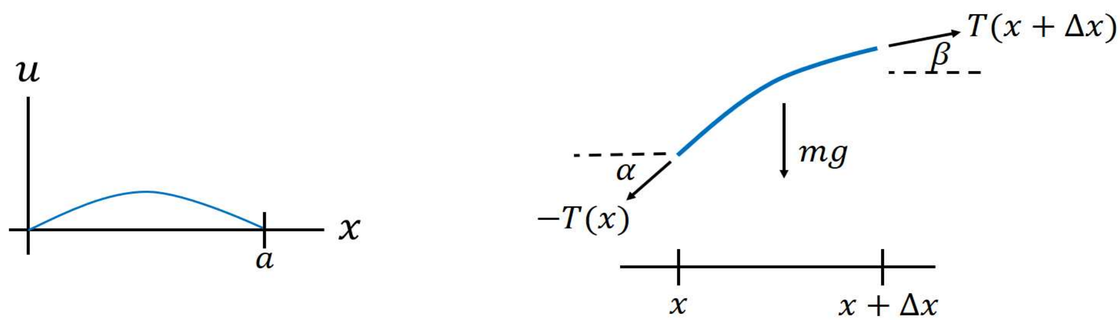
In fact, force and acceleration may be vectors, $\vec{F}$ and $\vec{a}$.

Note. Now suppose we have a string with ends fixed and that the string is displaced from the equilibrium. Suppose the string lies along the x-axis at equilibrium and that $u(x,t)$ represents the displacement of the string at spatial coordinate x and at time t. We make two assumptions:

(1) The string is perfectly flexible, i.e. it offers no resistance to bending. This implies that the tension T in the string at a given point is tangent to the string at the point.

(2) A point on the string moves only in the vertical direction.

We then have:



We let time be fixed and consider the forces on a segment of the string from x to $x+\Delta x$. The horizontal force at x due to tension is $-T(x)\cos\alpha$ and the horizontal force at $x+\Delta x$ due to tension is $T(x+\Delta x)\cos\beta$. By assumptions (1) and (2) and Newton's Second law of Motion,

$$-T(x)\cos\alpha + T(x+\Delta x)\cos\beta = 0.$$

By assumption (2), for fixed t, the horizontal component of tension is constant and so

$$T(x)\cos\alpha = T(x+\Delta x)\cos\beta = T(\text{contant})$$

and therefore

$$T(x) = \frac{T}{\cos\alpha} \text{ and } T(x+\Delta x) = \frac{T}{\cos\beta}. \qquad (*)$$

In the vertical direction we have a net force of

$$-T(x)\sin\alpha + T(x+\Delta x)\sin\beta - mg.$$

By Newton's Second Law,

$$-T(x)\sin\alpha + T(x+\Delta x)\sin\beta - mg = m \times \text{acceleration}.$$

Now the acceleration on this segment at x is $u_{tt}(x,t)$ and the acceleration at $x+\Delta x$ is $u_{tt}(x+\Delta x,t)$ (in our illustration, $u_{tt}(x+\Delta x,t) > u_{tt}(x,t)$) and by the Mean Value Theorem, the acceleration on this segment is $u_{tt}(k,t)$ for some $k \in [x, x+\Delta x]$. So we have

$$-T(x)\sin\alpha + T(x+\Delta x)\sin\beta - mg = mu_{tt}(k,t).$$

By $(*)$ we get

$$-T\tan\alpha + T\tan\beta - mg = mu_{tt}(k,t).$$

We now let $\rho = m/L$ be the *linear density* of the string (where m is the mass of a segment of the string of length L). Then for the segment x to $x+\Delta x$ we get (by assumption (2)) that $m = \rho\Delta x$ and so

$$-T\tan\alpha + T\tan\beta - \rho\Delta x g = \rho\Delta x u_{tt}(k,t). \qquad (**)$$

Now the slope of $u(x,t)$ (for fixed t) for a given x value in the xu-plane if $\partial u/\partial x = u_x$. Recall that slopes of tangent lines are also equal to tangents of angles of tangent lines. So $\tan(\alpha+\pi) = \tan\alpha = u_x(x,t)$ and $\tan\beta = u_x(x+\Delta x,t)$. So from $(**)$ we get

$$-Tu_x(x,t) + Tu_x(x+\Delta x,t) = \rho\Delta x u_{tt}(k,t) + \rho\Delta x g$$

for some $k \in [x, x + \Delta x]$. Therefore

$$T \frac{u_x(x + \Delta x, t) - u(x,t)}{\Delta x} = \rho(u_{tt}(k,t) + g).$$

Letting $\Delta x \to 0$ we get that $k \to x$ and

$$T u_{xx}(x,t) = \rho u_{tt}(x,t) + \rho g, \text{ or } u_{xx} = \frac{\rho}{T} u_{tt} + \frac{\rho}{T} g.$$

If we set $c^2 = T/\rho$ we get

$$u_{xx} = \frac{1}{c^2} u_{tt} + \frac{1}{c^2} g \text{ or } u_{tt} + g = c^2 u_{xx}.$$

Note. If we ignore the influence of gravity above then we get the *wave equation*

$$\boxed{u_{tt} = c^2 u_{xx}}$$

where $c = \sqrt{T/\rho}$. The term c is called the *wave speed*.

Note. We now consider variations of the 1-dimensional wave equation above. If we assume a resistance due to the medium containing the string which is proportional to velocity (a common assumption) then we get

$$u_{tt} - c^2 u_{xx} + R u_t = 0 \ (R > 0).$$

Of we assume a "transverse elastic force" (which is equivalent to assuming springs are attached to the string; recall Hooke's Law of the Spring: If a spring is displaced from equilibrium by an amount x, then a force of $F = kx$ is exerted by the spring; k is the "spring constant.") then we get

$$u_{tt} - c^2 u_{xx} + ku = 0 \ (k > 0).$$

An externally applied force can produce a nonhomogeneous PDE:

$$u_{tt} - c^2 u_{xx} = f(x,t).$$

Note. We temporarily postpone a dimension of the 2-dimensional (and higher dimensional) wave equation.

Note. We next consider the diffusion equation. We need to first recall the definition of the divergence of a vector valued function and the Divergence Theorem.

Note. Recall that the *divergence* of $\vec{v}(x,y,z)$ is

$$\mathrm{div}(\vec{v}(x,y,z)) = \mathrm{div}(\langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle) = M_x + N_y + P_z.$$

This is denoted $\nabla \cdot \vec{v}$.

Theorem. The Divergence Theorem.
Let T be a solid totally bounded by a closed surface S which consists of finitely many smooth prices. If the components of $\vec{v} = \vec{v}(x,y,z)$ are continuously differentiable on T, then

$$\int\int_S \vec{v}\cdot\vec{n}\,ds = \int\int\int_T \mathrm{div}(\vec{v})\,dx\,dy\,dz$$

where $\vec{n}$ is a unit vector normal to surface S.

Note. Consider a motionless fluid in a tube with a substance suspended in the fluid with concentration $u(x,t)$. Then the amount of suspended substance in the pipe from $x = x_0$ to $x = x_1$ is $M(t) = \int_{x_0}^{x_1} u(x,t)\,dx$ and so $\dfrac{dM}{dt} = \displaystyle\int_{x_0}^{x_1} u_t(x,t)\,dx$. Frick's Law of Diffusion states that the flow of the suspended substance is proportional to the concentration gradient ($u_x(x,t)$ in this case). So

$$\frac{dM}{dt} = \left(\begin{array}{c} \text{flow in pipe from} \\ x = x_0 \text{ to } x = x_1 \end{array} \right) - (\text{flow out}) = ku_x(x_1,t) = ku_x(x_0,t).$$

Therefore

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = ku_x(x,t) = ku_x(x_0,t)$$

and differentiating with respect to x_1 gives $u_t(x_1,t) = ku_{xx}(x_1,t)$. Since x_1 was arbitrary, we get

$$\boxed{u_t = ku_{xx}.}$$

This is the *one-dimensional diffusion equation.*

Note. In 3 dimensions we get that the rate of change of mass of suspended substance in a solid domain D with surface $S = \text{bdy}(D)$ is

$$\begin{aligned} \frac{dM}{dt} &= \int\int\int_D u_t\,dx\,dy\,dz \text{ (as above)} \\ &= \int\int_{\text{bdy}(D)} \vec{n}\cdot(k\nabla u)\,ds \text{ by Frick's Law and calculating flow through bdy}(D). \end{aligned}$$

By the Divergence Theorem,

$$\text{div}(k\nabla u) = \nabla\cdot(k\nabla u) = k\nabla^2 u = u_t$$

or

$$\boxed{u_t = k(u_{xx} + u_{yy} + u_{zz}).}$$

Note. Finally, we consider heat flow. Recall (maybe from Chemistry class) that the *heat capacity* of a solution is the amount of heat energy needed to raise one gram of the substance 1° C. The units are J/g °C). Let $u(x, y, z, t)$ be the temperature and let $H(t)$ be the amount of heat (in J) contained in a solid region D. Then

$$H(t) = \int\int\int_D c\rho u \, dx \, dy \, dz$$

where c is the specific heat and ρ is density of the material. As above,

$$\frac{dH}{dt} = \int\int\int_D c\rho u_t \, dx \, dy \, dz.$$

Fourier's Law (a spiced-up version of Newton's Law of Cooling) says that heat energy flows from hot to cold at a rate proportional to the temperature gradient, ∇u. So

$$\frac{dH}{dt} = \int\int_{\text{bdy}(D)} \vec{n} \cdot (\kappa \nabla u) \, ds$$

where κ is a constant (the "heat conductivity"). By the Divergence Theorem,

$$\nabla \cdot (\kappa \nabla u) = \kappa \nabla^2 u = c\rho u_t$$

or

$$\boxed{c\rho u_t = \kappa(u_{xx} + u_{yy} + u_{zz}).}$$

Notice that at equilibrium $u_t = 0$ and so $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$.

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