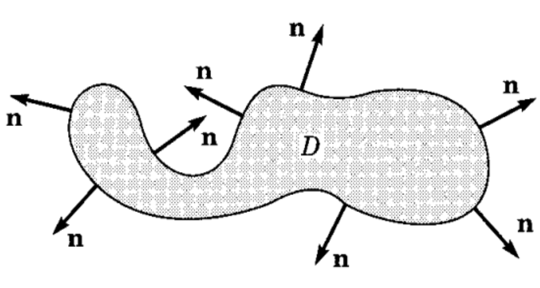
Section 1.4. Initial and Boundary Conditions

Note. In this section, we define initial conditions and three types of boundary conditions. We give an example of each of the boundary conditions.

Note/Definition. For a first order PDE involving $u(\vec{x}, t)$ where $\vec{x}$ is a vector of physical dimensions 1, 2, or 3, an *initial condition* is of the form $u(\vec{x}, t_0) = \varphi(\vec{x})$. If the PDE is of second order, we may have two *initial conditions* of the form

$$\begin{cases} u(\vec{x}, t_0) = \varphi(\vec{x}) \\ u_t(\vec{x}, t_0) = \psi(\vec{x}). \end{cases}$$

Note. If we have a 2-dimensional region in the plane with a smooth boundary, then we can talk about *unit normal vectors* $\vec{n}$ to the region (this is Figure 1 of Section 1.4):



If $u = u(x, y)$ then $\partial u/\partial n = \vec{b} \cdot \nabla u$ represents the flow of u through the boundary of the region. Similar statements can be made about 1 and 3 dimensional "regions."

Definition. If a PDE is defined on a domain D (an open connected set), then a condition on the unknown function u which involves the boundary of D is a *boundary condition.* The types of boundary conditions we will consider are

1. u is specified on bdy(D), called a *Dirichlet condition*,

2. the normal derivative $\partial u/\partial n$ is specified, called a *Neumann condition*,

3. $\dfrac{\partial u}{\partial n} + au$ is specified where $a = a(s, y, z, t)$, called a *Robin condition.*

If the specified quantity is identically 0, the boundary conditions are *homogeneous.* Otherwise the boundary conditions are inhomogeneous.

Note. We now give three examples of boundary conditions.

Note. Consider the 1-dimensional wave equation where the vibrating string is fixed at both ends. This translates as $u(0, t) = u(\ell, t) = 0$ where ℓ is the length of the string (notice the typo on page 21). This is an example of a Dirichlet condition.

Note. Consider the diffusion equation where the suspended substance is constrained to remain in a container. Since no substance can escape through the walls of the container $\partial u/\partial n = 0$. This is an example of a Neumann condition.

Note. Consider the 1-dimensional heat equation in a rod (with no loss of heat out of the rod except at its ends; i.e., the rod is insulated). If the end of the rod (at $x = \ell$) is immersed in a reservoir of temperature $g(t)$ and if we assume Newton's Law of Cooling, then at $x = \ell$ we get

$$\frac{\partial u}{\partial n}(\ell, t) = a(g(t) - u(\ell, t))$$

where $a < 0$ constant (heat is lost if $g(t) < u(\ell, t)$) and so

$$\frac{\partial u}{\partial n}(\ell, t) + au(\ell, t) = ag(t).$$

This is an example of a Robin condition.

Note. It is common to consider PDEs over infinite regions. For example, we might consider heat distribution in an infinite rod. We would then require that the temperature tend to 0 as the spatial variables tend to ∞ (or else the rod would contain an infinite amount of energy). This is an example of a *condition at infinity.*

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