Chapter 2. Waves and Diffusions

Note. In this chapter we study the wave and diffusion equations on $-\infty < x < \infty$ (this avoids the study of boundary conditions).

Section 2.1. The Wave Equation

Note. In this section, we state two theorems concerning the wave equation and give a technique to solve an associated IVP.

Theorem. The general solution of $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, is

$$u(x,t) = f(x+ct) + g(x-ct). \qquad (3)$$

That is, (3) is a solution of the PDE and every solution of the PDE is of the form (3) for some f and g. Notice that f and g must be twice differentiable.

Proof. Notice that we can factor the operator which represents the PDE:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

If we let $v = u_t + cu_x$ then it must be that

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = v_t - cv_x = 0.$$

So we have the system of PDEs (which is equivalent to the original ODE):

$$\begin{cases} v_t - cv_x = 0 \\ u_t + cv_x = v. \end{cases}$$

As we saw in Section 1.2, the solution to $v_t - cv_x = 0$ is $v(x,t) = h(x+ct)$ where h is an arbitrary differentiable function. So we now consider the nonhomogeneous ODE

$$u_t + cu_x = h(x+ct). \qquad (4c)$$

The general solution to the associated homogeneous PDE, $u_t + cu_x = 0$, is $g(x-ct)$. Now consider f defined such that $f' = h(2c)$. Then

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f(x+ct) = 2cf'(x+ct) = h(x+ct).$$

So f defined as $1/(2c)\int h$ (an anitiderivative of h) is a solution to the nonhomogeneous ODE (4c) (since h is arbitrary differentiable, f is arbitrary twice differentiable). Therefore, as with ODEs (the proof is similar and uses linear operators) the general solution to the given PDE is

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary twice differentiable functions. ∎

Theorem. Consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in \mathbb{R} \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

This IVP has a unique solution.

Proof. We have seen that the general solution of the PDE $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = f(x+ct) + g(x-ct).$$

With $t = 0$ we get $\varphi(x) = f(x) + g(x)$ and $\psi(x) = cf('(x) + xg'(x)$. Differentiating φ and dividing ψ by c gives

$$\varphi' = f' + g' \text{ and } \frac{1}{c}\psi = f' - g',$$

or

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f' \\ g' \end{bmatrix} = \begin{bmatrix} \varphi' \\ \frac{1}{2}\psi \end{bmatrix}.$$

This is a system of two unknowns in two equations and has the unique solution

$$f' = \frac{1}{2}\left(\varphi' + \frac{\psi}{c}\right) \text{ and } g' = \frac{1}{2}\left(\varphi' - \frac{\psi}{c}\right).$$

Integrating we get

$$f(s) = \frac{1}{2}\varphi(s) + \frac{1}{2s}\int_0^s \psi(x)\,dx + A \text{ and } g(s) = \frac{1}{2}\varphi(s) - \frac{1}{2s}\int_0^s \psi(x)\,dx + B$$

for constants A and B. Now

$$\varphi(x) = f(x) + g(x)$$

$$= \left(\frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi + A\right) + \left(\frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi + B\right) = \varphi(x) + A + B.$$

So we must have $A + B = 0$. so taking $s = x + ct$ in the formula for f and $s = x - ct$ in the formula for g gives

$$u(x,t) = f(x+ct) + g(x-ct)$$

$$= \left(\frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds + A\right) + \left(\frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds + B\right)$$

or

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

∎

Example. Page 35 Example 2. Consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in \mathbb{R} \\ u(x,0) = \varphi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| \geq a \end{cases} \\ u_(x,0) = \psi(x) = 0. \end{cases}$$

Find the solution. This is called the "three finger pluck."

Solution. The solution is

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right).$$

Now we get (see page 36 for some details; this is Figure 2.1.2 from page 36):