Section 2.2. Causality and Energy

Note. In this section, we define kinetic and potential energy for the wave equation and show that the total energy is constant in a wave equation model.

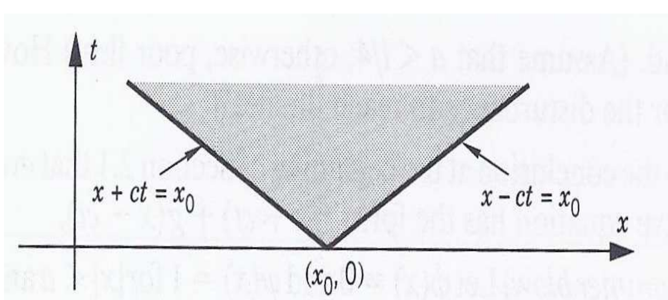
Note/Definition. Since the solution of the wave equation IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} \text{ for } x \in \mathbb{R} \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \end{cases}$$

is

$$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds,$$

if $\psi(s) = 0$ then the solution is a pair of waves, one moving to the left at a rate of c and one moving to the right at a rate of c. If initial velocity is not zero (i.e., $\psi(x) \not\equiv 0$), then we claim that $u(x,t)$ describes waves moving no faster than c. This is called the *Principle of Causality*. This has the following implication. If we consider the initial conditions at point $(x_0, 0)$, this can only have an effect on points in the *domain of influence* in the xt-plane of:



Note. Recall that the kinetic energy of a particle of mass m and velocity v is $K = \frac{1}{2}mv^2$. The potential energy in a conservative system if a quantity P where $K + P = \text{constant}$.

Definition. The *kinetic energy* in the wave equation is given by

$$K = \frac{1}{2}\rho \int_{-\infty}^{\infty} (u_t)^2 \, dx.$$

The *potential energy* is

$$P = \frac{1}{2}T \int_{-\infty}^{\infty} (u_x)^2 \, dx.$$

The *total energy* is $E = K + P$.

Theorem. In the wave equation, total energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) \, dx$$

is constant. (We assume $\varphi(x)$ and $\psi(x)$ vanish outside an interval $|x| \le R$.)

Proof. As argued above, the "domain of influence" of the interval $|x| \le R$ at time t is the interval $|x| \le R + ct$. Therefore $u(x,t)$ and $u_t(x,t)$ are 0 for $|x| > R + ct$ (at time t). Now

$$\frac{dK}{dt} = \frac{d}{dt}\left[\frac{1}{2}\rho \int (u_t)^2 \, dx\right] = \frac{1}{2}\rho \int 2u_t u_{tt} \, dx = \rho \int u_t u_{tt} \, dx.$$

Since $\rho u_{tt} = T u_{xx}$ (from the wave equation),

$$\frac{dK}{dt} = T \int u_t u_{xx} \, dx - \left(T u_t u_x - \int u_x T u_{tx} \, dx \right)\Bigg|_{-\infty}^{\infty}.$$

Now $u_t|_{-\infty}^{\infty} = 0$ so

$$\frac{dK}{dt} = -\int u_x T u_{tx}\, dx = -T \int \left(\frac{1}{2} u_x\right)_t \, dx = -\frac{d}{dt}\left[\frac{1}{2} T \int u_x^2 \, dx\right] = -\frac{dP}{dt}.$$

Therefore $\dfrac{dE}{dt} = \dfrac{dK}{dt} + \dfrac{dP}{dt} = 0$ and $E = K + P =$ constant. ∎

Note. The above theorem is called the *Law of Conservation of Energy*. This law can be used to prove the Principle of Causality.

Revised: 3/22/2019