Section 2.3. The Diffusion Equation

Note. Recall the Diffusion Equation: $u_t - ku_{xx}$, $k > 0$. In this section, we study some general properties of this equation and in the next section we find the general solution for $-\infty < x < \infty$.

Theorem. The Maximum Principle.
If $u(x,t)$ is a solution to the diffusion equation in the region $R = \{(x,t) \mid x \in [0,\ell], t \in [0,T]\}$ then the maximum value of $u(x,t)$ is assumed at either $t = 0$ or $x = 0$ or $x = \ell$ (i.e., on the boundary of the region).

Proof. Let M be the maximum of $u(x,t)$ for $t - 0$, $x - 0$, or $x = \ell$. Let $\varepsilon > 0$ and let $v(x,t) = u(x,t) + \varepsilon + x^2$. Since $0 \le x \le \ell$, $v(x,t) \le u(x,t) + \varepsilon\ell^2$ on R. Also

$$v_t - kv_{xx} = u_t - k(u + \varepsilon x^2_{xx} = u_t - ku_{xx} - 2k\varepsilon = -2k\varepsilon < 0. \qquad (2)$$

Now suppose $v(x,t)$ attains its maximum at $(x_0,t_0) \in \text{int}(R) = \{(x,t) \mid x \in (0,\ell), t \in (0,T)\}$. Then it must be that $\text{grad}(v) = \nabla v = v_x + v_t = 0$ at (x_0,t_0). Also, holding t constant at $t = t_0$ (taking a cross section), we need $u(x,t_0)$ to be concave down and so $u_{xx}(x_0,t_0) < 0$. However, this contradicts (2). So thee cannot be a maximum in $\text{int}(R)$. Now suppose v has a maximum on $\{(x,t) \mid x \in (0,\ell)\}$. Then $v_x(x_0,T) = 0$ and $v_{xx}(x_0,T) \le 0$. Now for $\delta > 0$ (where $T_0 - \delta > 0$) we have

$$v_t(x_0,T) = \lim_{\delta \to 0^+} \frac{v(x_0,T) - v(x_0,T_0 - \delta)}{\delta} \ge 0.$$

But then $v_t - kv_{xx} \ge 0$, contradicting (2).

Therefore the maximum of v occurs on either $t - 0$, $x = 0$, or $x = \ell$. Let S

denote this set $\{(x,t) \mid t = 0, \text{ or } x = 0, \text{ or } x = \ell\}$. Then

$$\max_R v(x,t) = \max_S v(x,t) = \max_S (u(x,t) + \varepsilon x^2) = M + \varepsilon \ell^2.$$

Therefore,

$$u(x,t) = v(x,t) - \varepsilon x^2 \le (M + \varepsilon \ell^2) - \varepsilon x^2 = M + \varepsilon(\ell^2 - x^2)$$

and so $u(x,t) \le M$ on R. ∎

Note/Definition. The above is the *weak version* of the Maximum Principle. The *strong version* states that the maximum which occurs on the boundary cannot also occur on the interior (the proof is much more difficult then the weak version).

Note. We can also prove that $u(x,t)$ attains its minimum on the boundary (applying the Maximum Principle to $-u(x,t)$). Next, for a uniqueness result.

Theorem. The nonhomogeneous Dirichlet problem for the diffusion equation,

$$u_t - k u_{xx} = f(x,t) \text{ for } 0 < x < \ell,\ t > 0$$
$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = g(t),\ u(\ell,t) = h(t) \end{cases}$$

has at most one solution (for the given functions f, φ, g, and h.

Proof. Suppose not, suppose u_1 and u_2 are both solutions. Consider $w = u_1 - u_2$. Then

$$w_t - k w_{xx} = ((u_1)_t - k(u_1)_{xx}) = ((u_2)_t - k(u_2)_{xx}) = f(x,t) = f(x,t) = 0$$

and

$$\begin{aligned} w(x,0) &= u_1(x,0) - u_2(x,0) = \varphi(x) = \varphi(x) = 0, \\ w(0,t) &= u_1(0,t) = u_2(0,t) = g(t) - g(t) = 0, \\ w(\ell,t) &= u_1(\ell,t) - u_2(\ell,t) = h(t) - h(t) = 0. \end{aligned}$$

So $w = 0$ on the boundary of $R = \{(x,t) \mid 0 \le x \le \ell,\, t \ge 0\}$ and by the Maximum Principle, $w(x,t) \le 0$ on region R. Similarly, by the Minimum Principle $w(x,t) \ge 0$ on R. Therefore $w = 0$ on R and $u_1 = u_2$. ∎

Note. We can also give a proof of the previous uniqueness theorem using the "energy method."

Proof. As above, let $w = u_1 - u_2$. Then

$$0 = 0w = (w_t - kw_{xx})w = \left(\frac{1}{2}w^2\right)_t + (-kw_x w)_x + kw_x^2$$

and so

$$0 = \int_0^\ell 0\,dx = \int_0^\ell \left(\frac{1}{2}w^2\right)_t dx - kw_x w\Big|_{x=0}^{x=\ell} + k\int_0^\ell w_x^2\,dx.$$

Now $w(0,t) = u_1(0,t) - u_2(0,t) = g(t) - g(t) = 0$ and $w(\ell,t) = u_1(\ell,t) - u_2(\ell,t) = h(t) - h(t) = 0$, so

$$\frac{d}{dt}\left[\int_0^\ell \frac{1}{2}w^2\,dx\right] = -k\int_0^\ell (w_x)^2\,dx \le 0 \text{ since } k > 0.$$

So $\int_0^\ell w^2\,dx$ is a decreasing function of time on

$$\int_0^\ell (w(x,t))^2\,dx \le \int_0^\ell (w(x,0))^2\,dx \qquad (4)$$

for $t \ge 0$. But $w(x,0) = u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0$ so $\int_0^\ell (w(x,t))^2\,dx = 0$ and $w(x,t) \equiv 0$. Therefore $u_1 x,t) = u_2(x,t)$. ∎

Note. Finally, we now address stability. We employ the "L^2 norm." For a square integrable function f on the interval $[0, \ell]$ we have the L^2 norm of f:

$$\|f\|_2 = \left\{ \int_0^\ell |f(x)|^2 \, dx \right\}^{1/2}.$$

Theorem. Consider the BVPs

$$(1) \qquad \begin{cases} u_t - ku_{xx} = f(x,t),\ 0 < x < \ell,\ t > 0 \\ u(x,0) = \varphi_1(x) \\ u(0,t) = g(t);\ u(\ell,t) = h(t) \end{cases}$$

and

$$(2) \qquad \{u(x,0) = \varphi_2(x).$$

If $u_1(x,t)$ is a solution to (1) and $u_2(x,t)$ is a solution to (2) then

$$\|u_1 - u_2\|_2 \le \|\varphi_1 - \varphi_2\|_2$$

for all t.

Proof. From (4) above we see that the result follows. ∎

Note. In fact, the previous result also holds for the "L^∞ norm." For continuous function f on the interval $[0, \ell]$ we have the L^∞ norm of f:

$$\|f\|_\infty = \max\{|f(x)| \mid x \in [0, \ell]\}.$$

Theorem. Consider the BVPs

$$(1) \qquad \begin{cases} u_t - ku_{xx} = f(x,t),\ 0 < x < \ell,\ t > 0 \\ u(x,0) = \varphi_1(x) \\ u(0,t) = g(t);\ u(\ell,t) = h(t) \end{cases}$$

and

$$(2) \qquad \{u(x,0) = \varphi_2(x).$$

If $u_1(x,t)$ is a solution to (1) and $u_2(x,t)$ is a solution to (2) then

$$\|u_1 - u_2\|_infty \leq \|\varphi_1 - \varphi_2\|_\infty$$

for all t.

Proof. If we have u_1 is a solution to (1) and u_2 a solution to (2) then $u_1 - u_2$ is a solution to the IVP

$$\begin{cases} u_t - ku_{xx} = f(x,t),\ 0 < x < \ell,\ t > 0 \\ u(x,0) = \varphi_1(x) - \varphi_2(x) \\ u(0,t) = u(\ell,t) = 0. \end{cases}$$

By the Maximum Principle,

$$u_1(x,t) - u_2(x,t) \leq \max|\varphi_1 - \varphi_2|$$

and by the Minimum Principle,

$$u_1(x,t) - u_2(x,t) \geq -\min|\varphi_1 - \varphi_2|$$

throughout the interior of region R. Therefore,

$$\|u_1 - u_2\|_\infty = \max_{0 \leq x \leq \ell} |u_1(x,t) - u_2(x,t)| \leq \max_{0 \leq x \leq \ell} |\varphi_1(x) = \varphi_2(x)| = \|\varphi_1 - \varphi_2\|_\infty$$

for $t > 0$. ∎