Section 2.4. Diffusion on the Whole Line

Note. In this section we solve an IVP for the diffusion equation $u_t = ku_{xx}$.

Note. We now solve the IVP

$$\begin{cases} u_t = ku_{xx},\ -\infty < x < \infty,\ t > 0 \\ u(x,0) = \varphi(x). \end{cases}$$

We need the following properties of the diffusion equation $u_t = ku_{xx}$:

(1) If $u(x,t)$ is a solution, then for any fixed y, $u(x-y,t)$ is also a solution.

(2) Any derivative of a solution is also a solution.

(3) Any linear combination of solutions if also a solution.

(4) If $S(x,t)$ is a solution, then so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any $g(y)$ provided the integral converges.

(5) If $u(x,t)$ is a solution, then so is $u(\sqrt{a}x, at)$ for any $a > 0$.

Lemma. A solution to

$$Q_t = kQ_{xx}$$
$$\begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$$

is

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp.$$

Proof. We look for a solution of the form $Q(x,t) = g(p)$ where $p = x/\sqrt{4kt}$ (motivated by (5) above). Now

$$\begin{aligned} Q_t &= -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) \\ Q_x &= \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{1}{4ht}g''(p) \end{aligned}$$

and so

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left(-\frac{1}{2}pg'(p) = \frac{1}{4}g''(p)\right)$$

or $g'' + 2pg' = 0$. Multiplying by the integrating factor e^{p^2}, we get

$$e^{p^2}g'' + 2pg' = 0,$$

or $e^{p^2}g' = \text{constant}$. Therefore $g(p) = c_1 \int e^{-p^2}\,dp + c_2$ and we choose

$$Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp + c_2.$$

Now to evaluate c_1 and c_2. If $x > 0$ and $t \to 0^+$, then $x/\sqrt{4kt} \to \infty$ and so

$$1 = \lim_{t\to 0^+} Q(x,t) = c_1 \int_0^{\infty} e^{-p^2}\,dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2.$$

If $x < 0$ and $t \to 0^+$, then $x/\sqrt{4kt} \to -\infty$ and so

$$0 = \lim_{t\to o^+} Q(x,t) = c_1 \int_0^{-\infty} e^{-p^2}\,dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2.$$

We get $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. So

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp.$$

∎

Theorem. The unique solution to

$$u_t = ku_{xx},\ -\infty < \infty,\ t > 0$$
$$u(x,0) = \varphi(x)$$

where $\varphi(x) = 0$ for $|x| > R$ for some R, is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} \varphi(y)\, dy.$$

Proof. With the notation of the above lemma, define

$$S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/(4kt)},\ t > 0.$$

By property (2), S is a solution to the diffusion equation. Also

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\, dy,\ t > 0$$

is a solution by property (2) above. Now we need only show $u(x,0) = \varphi(x)$. Well,

$$u(x,t) = \int_{\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\varphi(y)\, dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\varphi(y)\, dy$$

$$= -\left(\varphi(y)Q(x-y,t) - \int Q(x-y,t)\varphi'(y)\, dy\right)\Bigg|_{-\infty}^{\infty} = \int_{-\infty}^{\infty} Q(x-y,t)\varphi'(y)\, dy.$$

So

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\varphi'(y)\, dy \\ &= \int_{-\infty}^{x} \varphi'(y)\, dy \text{ since} Q(x-y,0) = 0 \text{ for } x-y<0 \text{ or } y > x \\ &= \varphi(y)_{-\infty}^{x} = \varphi(x). \end{aligned}$$

So

$$\int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\, dy = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi kt}} e^{-(x-y)^2/(4kt)} \varphi(y)\, dt.$$

The result follows. ∎

Definition. The *error function* is

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp.$$

Note. We have

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathrm{Efr}\left(\frac{x}{\sqrt{4kt}}\right).$$

Example. Page 52 Number 16. Solve the IVP:

$$\begin{cases} u_t - ku_{xx} + bu = 0, \ -\infty < x < \infty \\ u(x,0) = \varphi(x). \end{cases}$$

Solution. We let $u(x,t) = e^{-bt}v(x,t)$. Then

$$\begin{aligned} u_t &= -be^{-bt}v(x,t) + e^{-bt}v_t(x,t) \\ u_{xx} &= e^{-bt}v_{xx}(x,t). \end{aligned}$$

So

$$u_t - ku_{xx} + bu = (-be^{-bt}v(x,t) + e^{-bt}v_t(x,t)) - ke^{-bt}v_{xx}(x,t) + be^{-bt}v(x,t) = 0,$$

which implies

$$\begin{cases} v_t - kv_{xx} = 0, \ -\infty < x < \infty \\ v(x,0) = u(x,0) = \varphi(x). \end{cases}$$

So from the first theorem of this section

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} \varphi(y)\, dy$$

and so the solution to the given IVP is

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} \varphi(y)\, dy.$$

□

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