Chapter 4. Boundary Problems

Section 4.1. Separation of Variables, The Dirichlet Condition

Note. In this section, we state two boundary value problems ("BVPs") which can be solved by separation of variables.

Note. The Wave Equation Dirichlet Problem is:

$$u_{tt} = c^2 u_{xx},\ 0 < x < \ell,\ t > 0$$
$$\begin{cases} u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases}$$

has solution (found by separation of variables)

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{\ell} + B_n \sin \frac{n\pi ct}{\ell} \right) \sin \frac{n\pi x}{\ell}$$

provided

$$\begin{aligned} \varphi(x) &= \sum_n A_n \sin \frac{n\pi x}{\ell} \\ \psi(x) &= \sum_n \frac{n\pi c}{\ell} B_n \sin \frac{n\pi x}{\ell}. \end{aligned}$$

These sums may be infinite and are, in that case, called *Fourier series.*

Note. The Diffusion Equation Dirichlet Problem is

$$u_t = ku_{xx},\ 0 < x < \ell,\ t > 0$$
$$\begin{cases} u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \varphi(x), \end{cases}$$

has solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \sin \frac{n\pi x}{\ell}$$

provided

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}.$$

See the comments on page 86.

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