Section 4.2. The Neumann Condition

Note. In this section, we consider the Neumann boundary condition for the wave equation.

Note. The Wave Equation Neumann Problem is:

$$\left\{ \begin{array}{l} u_{tt} = c^2 u_{xx},\ 0 < x < \ell,\ t > 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \end{array} \right.$$

has general solution

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \cos\frac{n\pi ct}{\ell}.$$

If, in addition,

$$\varphi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell}$$

then the initial condition $u(x,0) = \varphi(x)$ is also satisfied. The solution with added initial condition $u_t(x,0) = \psi(x)$ is

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi ct}{\ell} + B_n \sin\frac{n\pi ct}{\ell} \right) \cos\frac{n\pi x}{\ell}$$

provided

$$\begin{aligned} \varphi(x) &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell} \\ \psi(x) &= \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} B_n \cos\frac{n\pi x}{\ell}. \end{aligned}$$

Example. Page 90 Number 1. Solve

$$\left\{\begin{array}{l} u_t = ku_{xx},\ 0 < x < \ell,\ t > 0 \\ u(0,t) = u_x(\ell,t) = 0. \end{array}\right.$$

Solution. Let's try separation of variables and consider $u(x,t) = X(x)T(t)$. Then the PDE implies

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda = \text{constant}$$

since the $T'/(kT)$ is a function of t only and X''/X is a function of x only. Then $T' = -\lambda kT$ which implies $T(t) = Ae^{-\lambda kt}$. Now the boundary condition $u(0,t) = 0$ implies $u(0,t) = X(0)T(t) = X(0)Ae^{-\lambda kt} = 0$ so that $X(0) = 0$. Next, $X'' = -\lambda X$ implies $X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x$ and the condition $X(0) = 0$ implies $C = 0$, so that $X(x) = D\sin\sqrt{\lambda}x$. The boundary condition $u_x(\ell,t) = 0$ implies $u_x(\ell,t) = \sqrt{\lambda}D\cos\sqrt{\lambda}\ell = 0$ so that we need $\sqrt{\lambda}\ell = (n+1/2)\pi$ or $\lambda = (n+1/2)^2\pi^2/\ell^2$ where $n \in \mathbb{N}$. Therefore $X(x) = D_n\sin((n+1/2)\pi/\ell x)$ where $n \in \mathbb{N}$. So we can take for any $n \in \mathbb{N}$

$$X(x)T(t) = D_n \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell}\right) A_n e^{-(n+1/2)^2\pi^2 kt/\ell^2}$$

and since sums of solutions are solutions, then

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell}\exp\left(-\left(n+\frac{1}{2}\right)^2\frac{\pi^2 kt}{\ell^2}\right),$$

where we have let A_n "absorb" D_n.