Section 4.3. The Robin Condition

Note. In this section, we consider Robin boundary condition for the wave equation and diffusion equation. We use separation of variables and have already observed that we know $T(t)$ from Section 4.1 in both cases (for given values of "eigenvalue" λ).

Note. We consider the homogeneous Robin condition for the wave or the diffusion equation. Separation of variables leads us to

$$\begin{cases} u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(\ell,t) + a_\ell u(\ell,t) = 0 \end{cases}$$

or

$$\begin{cases} X'(0) - a_0 X(0) = 0 \\ X'(\ell) + a_\ell X(\ell) = 0. \end{cases} \qquad (*)$$

We must deal with the ODE $X'' = -\lambda X$ with the above boundary conditions $(*)$. We are lead to 2 cases based on the sign of λ.

Note. If the eigenvalue λ is positive, suppose $\lambda = \beta^2$ where $\beta > 0$. Then $X'' = -\lambda X$ implies $X(x) = C\cos\beta x + D\sin\beta x$. The boundary condition at $x = 0$ gives

$$X'(0) - a_0 X(0) = \beta D - a_0 C = 0. \qquad (1)$$

The boundary condition at $x = \ell$ gives

$$X'(\ell) + a_\ell X(\ell) = (\beta D + a_\ell C)\cos\beta\ell + (-\beta C + a_\ell D)\sin\beta\ell = 0. \qquad (2)$$

We therefore have two linear equation sin two unknowns C and D. From (1) we see that $D = a_0 C/\beta$. Plugging this in to (2) gives

$$(a_0 C + a_\ell C)\cos\beta\ell + \left(-\beta C + \frac{a_\ell a_0 C}{\beta}\right)\sin\beta\ell = 0.$$

Dividing by $X\cos\beta\ell$ (assuming $C \neq 0$ and $\cos\beta\ell \neq 0$) gives the relationship

$$(\beta^2 - a_0 a_\ell)\tan\beta\ell = (a_0 + a_\ell)\beta.$$

So if β is a solution to this transcendental equation, then

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right)$$

is a solution to the ODE with given boundary conditions.

Note. If the eigenvalue λ is negative, suppose $\lambda = -\gamma^2$ where $\gamma > 0$. Then $X'' = -\lambda X$ has general solution

$$X(x) = Ae^{\gamma x} + Be^{-\gamma x} = C\cosh\gamma x + D\sinh\gamma x.$$

Applying the boundary conditions gives

$$\tanh\gamma\ell = -\frac{(a_0 + a_\ell)\gamma}{\gamma^2 + a_0 a + \ell}$$

which can be solved for the eigenvalues λ and the corresponding eigenfuctions are

$$X(x) = \cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x.$$

Note. Now we have $T(t)$ for the wave and diffusion equations in Section 4.1. So the "grand conclusion" for the Robin boundary conditions is that the wave and diffusion equations with boundary conditions

$$\begin{cases} u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(\ell,t) + a_\ell u(\ell,t) = 0 \end{cases}$$

has solution

$$u(x,t) = \sum_n T_n(t) X_n(x)$$

where $X_n(x)$ are eigenfunctions and

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n k t} \text{ for the diffusion equation} \\ A_n \cos\sqrt{\lambda_n} ct + \beta_n \sin\sqrt{\lambda_n} ct \text{ for the wave equation.} \end{cases}$$

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