Chapter 5. Fourier Series

Section 5.1. The Coefficients

Note. In this section, we define Fourier series and give some results and examples.

Definition. A *Fourier sine series* in the interval $(0, \ell)$ is a series of the form

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell}.$$

Note. It is shown in Calculus 2 (using summation an difference equations) that for $m, n \in \mathbb{N}$

$$\int_0^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell} dx = 0$$

if $m \neq n$.

Theorem. If $\varphi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/\ell)$ then

$$A_m = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin \frac{m\pi x}{\ell}\, dx.$$

Proof. Fix m and consider

$$\begin{aligned} \int_0^{\ell} \sin \frac{m\pi x}{\ell} \varphi(x)\, dx &= \int_0^{\ell} \left(\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} \right) \sin \frac{m\pi x}{\ell} \\ &= \sum_{n=1}^{\infty} A_n \int_0^{\ell} \sin \frac{n\pi x}{\ell} \sin \frac{m\pi x}{\ell}\, dx \text{ (TRUST ME!)} \\ &= A_m \int_0^{\ell} \sin^2 \frac{m\pi x}{\ell}\, dx = \frac{l}{2} A_m, \end{aligned}$$

and the claim follows. ∎

Note. The solution to the Dirichlet diffusion boundary problem

$$u_t = ku_{xx},\ 0 < x < \ell,\ t > 0$$
$$\begin{cases} u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \sin\frac{n\pi x}{\ell} \text{ where } A_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\sin\frac{n\pi x}{\ell}\,dx.$$

Note. The solution to the Dirichlet wave boundary value problem

$$u_{tt} = c^2 u_{xx},\ 0 < x < \ell$$
$$\begin{cases} u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{\ell} + B_n\sin\frac{n\pi ct}{\ell}\right)\sin\frac{n\pi x}{\ell}$$

where

$$A_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\sin\frac{n\pi x}{\ell}\,dx \text{ and } B_n = \frac{2}{n\pi c}\int_0^\ell \psi(x)\sin\frac{n\pi x}{\ell}\,dx.$$

Note. We now consider Fourier cosine series. As with sine functions, for $m, n \in \mathbb{N}$

$$\int_0^\ell \cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx = 0$$

if $m \neq n$. From this we get:

1. The Neumann diffusion boundary problem

$$u_t = ku_{xx},\ 0 < x < \ell,\ t > 0$$
$$\begin{cases} u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

has solution

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/\ell)^2 kt} \cos\frac{n\pi x}{\ell} \text{ where } A_n = \frac{2}{\ell}\int_0^{\ell} \varphi(x)\cos\frac{n\pi x}{\ell}\,dx,$$

and $n \geq 0$.

2. The Neumann wave boundary problem

$$u_{tt} = c^2 u_{xx},\ 0 < x < \ell$$
$$\begin{cases} u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

has solution

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{\ell} + B_n\sin\frac{n\pi ct}{\ell}\right)\cos\frac{n\pi x}{\ell}$$

where

$$A_n = \frac{2}{\ell}\int_0^{\ell}\varphi(x)\cos\frac{n\pi x}{\ell}\,dx \text{ and } B_n = \frac{2}{n\pi c}\int_0^{\ell}\psi(x)\cos\frac{n\pi x}{\ell}\,dx, n \geq 0.$$

Definition. The *(full) Fourier series* of $\varphi(x)$ in the interval $-\ell < x < \ell$ is

$$\varphi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{\ell} + B_n\sin\frac{n\pi x}{\ell}\right).$$

Theorem. If φ has a Fourier series as given above, then

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(x)\cos\frac{n\pi x}{\ell}\,dx \text{ and } B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(x)\sin\frac{n\pi x}{\ell}\,dx.$$

Proof. Let's introduce the notation

$$\varphi(x)\cdot f(x) = \frac{1}{\ell}\int_{-\ell}^{\ell} \varphi(x)f(x)\,dx.$$

Then notice

$$\cos\frac{n\pi x}{\ell}\cdot\sin\frac{m\pi x}{\ell} = 0 \text{ for all } n, m \in \mathbb{N}$$
$$\cos\frac{n\pi x}{\ell}\cdot\cos\frac{m\pi x}{\ell} = \sin\frac{n\pi x}{\ell}\cdot\sin\frac{m\pi x}{\ell} = 0 \text{ for } n \neq m$$
$$1\cdot\cos\frac{n\pi x}{\ell} = 1\cdot\sin\frac{n\pi x}{\ell} = 0,$$
$$\cos\frac{m\pi x}{\ell}\cdot\cos\frac{n\pi x}{\ell} = \sin\frac{n\pi x}{\ell}\cdot\sin\frac{n\pi x}{\ell} = 1$$
$$1\cdot 1 = 2.$$

So

$$\begin{aligned} A_n &= \varphi(x)\cdot\cos\frac{n\pi x}{\ell} = \frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\cos\frac{n\pi x}{\ell}\,dx \\ B_n &= \varphi(x)\cdot\sin\frac{n\pi x}{\ell} = \frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\sin\frac{n\pi x}{\ell}\,dx. \end{aligned}$$

∎

Example. Page 108 Number 4a. Find the Fourier cosine series for $|\sin x|$ on $(-\pi, \pi)$.

Solution. We try first to find a series on $(0, \pi)$. We want

$$|\sin x| = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{\pi},$$

so we need

$$A_0 = \frac{2}{\pi}\int_0^\pi |\sin x| \cos nx\, dx.$$

With $n = 0$ we have

$$A_0 = \frac{2}{\pi}\int_0^\pi \sin x\, dx = -\frac{2}{\pi}\cos x\Big|_0^\pi = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}.$$

For general $n \in \mathbb{N}$ we recall a trig identity:

$$\sin A \cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B).$$

Then

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx\, dx = \frac{2}{\pi}\int_0^\pi \left(\frac{1}{2}\sin(1-n)x + \frac{1}{2}\sin(n+1)x\right) dx \\ &= \frac{1}{\pi}\frac{-1}{1-n}\cos(1-n)x + \frac{1}{\pi}\frac{-1}{n+1}\cos(n+1)x\Big|_0^\pi \\ &= \left(\frac{1}{\pi(n-1)} - \frac{1}{\pi(n+1)}\right) \times \begin{cases} +1-1 \text{ if } n \text{ is odd} \\ -1-1 \text{ if } n \text{ is even} \end{cases} \\ &= \begin{cases} -\frac{2}{\pi}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

So

$$|\sin x| = \frac{2}{\pi} + \sum_{\substack{n=1 \\ n \text{ even}}}^\infty -\frac{4}{\pi}\frac{1}{n^2-1}\cos nx = \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ even}}}^\infty \frac{1}{1-n^2}\cos nx.$$

Notice this series is "even" and so is valid for $(-\pi, \pi)$. □

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