Section 5.2. Even, Odd, Periodic, and Complex Functions

Note. Notice that the text discusses several easy properties of even and odd functions on pages 109-111. We give a related definition and result.

Definition. The *periodic extension* of a Fourier series of φ valid on $-\ell < x < \ell$ is $\varphi_{\text{per}}(x) = \varphi(x - 2\ell m)$ for $2\ell m - \ell < x < 2\ell m + \ell$ and $m \in \mathbb{Z}$.

Theorem. If φ has a full Fourier series on $(-\ell, \ell)$ then

$$\varphi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} \varphi(x) e^{-in\pi x/\ell}\, dx.$$

Proof. First notice that

$$\int_{-\ell}^{\ell} e^{in\pi x/\ell} e^{-im\pi x/\ell}\, dx = 0 \text{ for } m \neq n$$

and

$$\int_{-\ell}^{\ell} e^{in\pi x/\ell} e^{-in\pi x/\ell}\, dx = 2\ell.$$

So

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} \varphi(x) e^{-in\pi x/\ell}\, dx = \frac{1}{2\ell} \int_{-\ell}^{\ell} \left(\sum_{m=-\infty}^{\infty} c_n e^{im\pi x/\ell} \right) e^{-in\pi/\ell}\, dx = \frac{1}{2\ell} \int_{-\ell}^{\ell} c_n\, dx = c_n.$$

∎

Note. The above theorem is just a reformulation of the full Fourier series based on the identities

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ and } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Revised: 3/23/2019