Section 5.3. Orthogonality and General Fourier Series

Note. In this section, we consider inner products and give a definition of symmetric boundary conditions. We then state and prove some theorems concerning symmetric boundary conditions.

Definition. The *inner product* of two functions f and g on $[a, b]$ is

$$(f, g) = \int_a^b f(x)g(x)\,dx.$$

If $(f, g) = 0$ then f and g are *orthogonal.*

Note. For operator A we have the boundary conditions:

1. Dirichlet: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$.

2. Neumann: $X_1'(a) = X_1'(b) = X_2'(a) = X_2'(b) = 0$.

3. Periodic: $X_1(a) = X_1(b)$, $X_2(a) = X_2(b)$, $X_1'(a) = X_1'(b)$, $X_2'(a) = X_2'(b)$.

4. Robin: $X_1'(a) + \alpha X_1(a) = 0 = X_1'(b) + \beta X_1(b)$, $X_2'(a) + \alpha X_2(a) = 0 = X_2'(b) + \beta X_2(b)$.

In each case, we can verify that if $\lambda_1 \neq \lambda_2$, then

$$(\lambda_2 - \lambda_1)(X_1, X_2) = (\lambda_2 - \lambda_1)\int_a^b X_1 X_2\,dx = \int_a^b ((-\lambda_1 X_1)X_2 + X_1(\lambda_2 X_2))\,dx$$

$$= \int_a^b (-X_1'' X_2 + X_1 X_2'')\,dx = \int_a^b (-X_1' X_2 + X_1 X_2')'\,dx.$$

Definition. Boundary conditions for an ODE on $[a,b]$ with eigenfunctions X_1 and X_2 are *symmetric* if

$$(X_2'(x)X_1(x) - X_2(x)X_1'(x))\big|_a^b = 0.$$

Note. Each of the boundary conditions above are symmetric.

Theorem 1(a). Symmetric boundary conditions implies that eigenfunctions of $A = d^2/dx^2$ corresponding to distinct eigenvalues are orthogonal.

Proof. Let X_1 and X_2 be eigenfunctions of A with corresponding eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$. Then $-X_1''X_2 + X_1X_2'' = (-X_1' + X_1X_2')'$ and so

$$(\lambda_2 - \lambda_2)(X_1, X_2) = \int_a^b (-X_1''X_2 + X_1X_2'')\,dx = (-X_1'X_2 + X_1X_2')\big|_a^b.$$

This is called *Green's second identity.* The claim follows. ∎

Theorem 1(b). With the hypotheses of Theorem 1(a), any function expanded in a series of the eigenfunctions has coefficients uniquely determined.

Proof. Let X_n be the eigenfunctions with eigenvalues λ_n. If $\varphi(x) = \sum_{n=1}^{\infty} A_nX_n(x)$, then

$$(\varphi, X_m) = \left(\sum_{n=1}^{\infty} A_nX_n, X_m\right) = \sum_{n=1}^{\infty}(A_nX_n, X_m) = A_m(X_m, X_m).$$

So $A_m = (\varphi, X_m)/(X_m, X_m)$ is uniquely determined, as claimed. ∎

Note. If there are two linearly independent function sX_1 and X_2 corresponding to eigenvalues $\lambda_1 = \lambda_2$, then by the Gram-Schmidt process, we can product two orthogonal functions X_1^* and X_2^* corresponding to $\lambda_1 = \lambda_2$.

Note. We now consider complex eigenfunctions. If f and g are complex valued functions of a real variable, then define the inner product

$$(f, g) = \int_a^b f(x)\overline{g(x)}\,dx.$$

Theorem 2. With symmetric boundary conditions, all eigenvalues are real and furthermore, all eigenfunctions can be chosen to be real valued.

Proof. Let λ be an eigenvalue of the operator $A = d^2/dx^2$ and let $X(x)$ be a corresponding eigenfunction. Then $X'' = \lambda X$ and $\overline{X}'' = \overline{\lambda}\,\overline{X}$. So $\overline{\lambda}$ is also an eigenvalue and by Green's second identity we have

$$\in_a^b (-X''\overline{X} + X\overline{X}'')\,dx = (-X'\overline{X} + X\overline{X}')\Big|_a^b = 0,$$

since the boundary conditions are symmetric. So $(\lambda - \overline{\lambda}) \int_a^b X\overline{X}\,dx = 0$, but $X\overline{X} = |X|^2 \geq 0$ and since X is an eigenfunction, $X \not\equiv 0$. Therefore $\lambda = \overline{\lambda}$ and so λ is real, as claimed. Next, if $X'' = \lambda X$ and $X = Y + iZ$, then $\overline{X}$ is also an eigenfunction. We can "replace" X and $\overline{X}$ with Y and Z (see page 117). ∎

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