Section 5.4. Completeness

Note. In this section, we explore some of the theory of Fourier series and state several theorems (and prove a few of them).

Definition. The series $\sum_{n=1}^{\infty} f_n(x)$ *converges* to $f(x)$ *pointwise* on (a,b) if for each $x^* \in (a,b)$ we have $\lim_{N\to\infty} \sum_{n=1}^{N} f(x^*) = f(x^*)$.

Definition. The series $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly* on $[a,b]$ if

$$\lim_{N\to\infty} \left(\max_{a\le x\le b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \right) = 0.$$

Definition. The series $\sum_{n=1}^{\infty} f_n(x)$ *converges in the L^2 sense* on (a,b) if

$$\lim_{N\to\infty} \left(\int_a^b \left| f(x) = \sum_{n=1}^{N} f_n(x) \right|^2 dx \right) = 0.$$

Theorem 1. There are an infinite number of eigenvalues for d^2/dx^2 with any symmetric boundary conditions. They form a sequence satisfying $\lambda_n \to \infty$.

Definition. The *Fourier coefficients* for $f(x)$ on (a,b) in terms of eigenfunctions $X_1, X_2, \ldots$ are

$$A_n = \frac{\int_a^b f(x)\overline{X_n(x)}\,dx}{\int_a^b |X_n(x)|^2\,dx}$$

and the Fourier series is $\sum_{n=1}^{\infty} A_n X_n(x)$.

Theorem 2. Uniform Convergence Theorem.

The series $\sum_{n=1}^{\infty} A_n X_n(x)$ converges to f uniformly on $[a,b]$ if

1. f, f', and f'' exist and are continuous for $a \le x \le b$, and

2. f satisfies the given symmetric boundary conditions.

Theorem 3. L^2 Convergence Theorem.

The series $\sum_{n=1}^{\infty} A_n X_n(x)$ converges to f in the L^2 sense in (a,b) if and only if f is any function for which

$$\int_a^b |f(x)|^2\,dx < \infty.$$

Note. Theorem 3 holds for Lebesgue integrals.

Theorem 4. Pointwise Convergence of Classical Fourier Series.

(i) The classical Fourier series (i.e., full, since, or cosine) converges to f pointwise on (a,b) if f is continuous on $a \le x \le b$ and f' is piecewise continuous on $a \le x \le b$.

(ii) If f and f' are piecewise continuous on $a \le x \le b$ and then the classical Fourier series converges at every point of $\mathbb{R}$ and the sum is

$$\sum_{n=1}^{\infty} = \frac{1}{2}(f(x^+) + f(x^-)) \text{ for } a < x < b$$

and converges to $\frac{1}{2}(f_{\text{ext}}(x^+) + f_{\text{ext}}(x^-))$ for all $x \in \mathbb{R}$ where where f_{ext} is the extended function (periodic, odd periodic, or even periodic).

Theorem 5. Let $\{X_n\}$ be any orthogonal set of functions. Let $\|f\|_2 < \infty$. Let N be a given positive integer. Among all possible choices of N constants $c_1, c_2, \ldots, c_N$, the choice that minimizes $\left\| f - \sum_{n=1}^{N} c_n x_n \right\|_2$ is c_i as the Fourier coefficients.

Proof. Let

$$E_n = \left\| f - \sum_{n=1}^{N} c_n X_n \right\|_2^2 = \int_a^b \left| f - \sum_{n=1}^{N} c_n X_n \right|^2 dx$$

$$= \int_a^b |f|^2 - 2 \sum_{n=1}^{N} c_n \int_a^b f X_n + \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m \int_a^b X_n X_m$$

$$= \|f\|_2^2 - 2 \sum_{n=1}^{N} c_n (f, X_n) + \sum_{n=1}^{N} c_n^2 \|X_n\|^2 = \sum_{n=1}^{N} \|X_n\|_2^2 \left(c_n - \frac{(f, X_n)}{\|X_n\|_2^2} \right)^2 + \|f\|_2^2$$

and E_n is minimal when $c_n = (f, X_n)/\|X_n\|_2^2$, as claimed. ∎

Corollary. Bessel's Inequality.

With the above notation,

$$\sum_{n=1}^{\infty} A_n \|X_n\|_2^2 \leq \|f\|_2^2.$$

Proof. With $c_n = A_n$ above, we get

$$E_n = \|f\|_2^2 - \sum_{n=1}^{N} \frac{(f, X_n)}{\|X_n\|_2^2} = \|f\|_2^2 = \sum_{n=1}^{N} A_n^2 \|X_n\|_2^2$$

and so

$$\sum_{n=1}^{N} A_n^2 \|X_n\|_2^2 \leq \|f\|_2^2$$

for all N and the claim holds. ∎

Theorem 6. Parseval's Equality.

The Fourier series of f converges to f in the L^2 sense if and only if

$$\sum_{n=1}^{\infty} |A_n|^2 \|X_n\|_2^2 = \|f\|_2^2.$$

Definition. The orthogonal set of functions $\{X_1, X_2, \ldots\}$ is *complete* if Parseval's Equality holds for all f with $\int_a^b |f|^2\,dx < \infty$.

Corollary 7. If $\int_a^b |f|^2 < \infty$ then Parseval's Equality holds.

Example. Page 131 Number 12. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution. Well,

$$x = \frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{\ell}$$

(see Page 106 Example 3) and so by Parseval's Equality

$$\int_0^\ell |x|^2\,dx = \sum_{n=1}^{\infty} \left|\frac{2\ell}{\pi n}\right|^2 \int_0^\ell \sin^2 \frac{n\pi x}{\ell}\,dx.$$

With $\ell = 1$,

$$\int_0^1 x^2\,dx = \sum_{n=1}^{\infty} \left(\frac{e}{n\pi}\right)^2 \int_0^1 \sin^2 n\pi x\,dx$$

or

$$\frac{1}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 \frac{1-\cos 2n\pi x}{2}\,dx = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{x}{2} - \frac{1}{2n\pi} \sin 2n\pi x\right)\Bigg|_0^1$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{2} - 0\right) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. $\square$