Chapter 6. Harmonic Functions

Section 6.1. Laplace's Equation

Note. In this section, we explore a common second order PDE and state some theorems related to it.

Note. At equilibrium in the wave and diffusion equations, $u_t = 0$ and these equations reduce to the *Laplace equation*

$$\begin{aligned} u_{xx} &= 0 \\ \Delta u = u_{xx} + u_{yy} &= 0 \\ \Delta u = u_{xx} + u_{yy} + u_{zz} &= 0. \end{aligned}$$

Definition. A solution to Laplace's equation is said to be *harmonic*.

Definition. *Poisson's equation* is the nonhomogeneous version of Laplace's equation: $\Delta u = f$.

Theorem. The Maximum Principle.
Let D be a connected bounded open set (in $\mathbb{R}^2$ or $\mathbb{R}^3$). Let $u(x, y)$ (or $u(x, y, z)$) be harmonic in D and continuous in $\overline{D} = D \cup \text{bdy}(D)$. Then the maximum and minimum values of n are attained on $\text{bdy}(D)$ and nowhere in D unless u is constant.

Theorem. The Uniqueness of Solutions to the Dirichlet Problem.
The Dirichlet problem

$$\begin{aligned} \Delta u &= f \text{ in open connected } D \\ u &= h \text{ on the boundary of } D \end{aligned}$$

has at most one solution.

Proof. Suppose not, suppose both u and v are solutions. Then $u - v$ is a solution to

$$\begin{aligned} \Delta u &= 0 \text{ in open connected } D \\ u &= 0 \text{ on the boundary of } D \end{aligned}$$

and so by the Maximum Principle (since $u - v$ is harmonic), the maximum and minimum of $u - v$ both occur on bdy(D) and are therefore 0. So $u = v$. ∎

Definition. A *translation* in $\mathbb{R}^2$ is a transformation

$$\begin{aligned} x' &= x + a \\ y' &= y + b. \end{aligned}$$

A *rotation* in $\mathbb{R}^2$ through angle α is a transformation

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned}$$

Theorem. Invariance in 2-Dimensions.

The Laplace Equation is invariant under translations and rotations in $\mathbb{R}^2$ (i.e., $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$).

Note. The proof of the above theorem is given on pages 150 and 151. In the proof, it is shown that in polar coordinates (r, θ) the Laplace operator satisfies:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Example. Page 154 Number 5. Solve

$$\begin{cases} u_{xx} + u_{yy} = 1 \text{ for } r < a \\ u = 0 \text{ for } r = a, \end{cases}$$

where r represents a distance from the origin (so r is a polar coordinate).

Solution. We search for a "rotationally invariant" solution (i.e., one independent of θ). In polar coordinates, the PDE becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 1 \text{ or } u_{rr} + \frac{1}{r}u_r = 1 \text{ or } ru_{rr} + u_r = r.$$

By the Product Rule, this becomes $(ru_r)_r = r$ and so $ru_r = \frac{1}{2}r^2 + c_1$ or $u_r = \frac{1}{2}r + c_1/r$ and hence $u = \frac{1}{4}r^2 - c_1 \log r + c_2$. Since $u = 0$ for $r = a$ we require $\frac{1}{4}a^2 - c_1 \log a + c_2 = 0$, so we get $c_2 = c_1 \log a - \frac{1}{4}a^2$. So the solution in terms of parameter c_1 is

$$u(r, \theta) = \frac{1}{4}r^2 - c_1 \log r + c_1 \log a - \frac{1}{4}a^2.$$

□