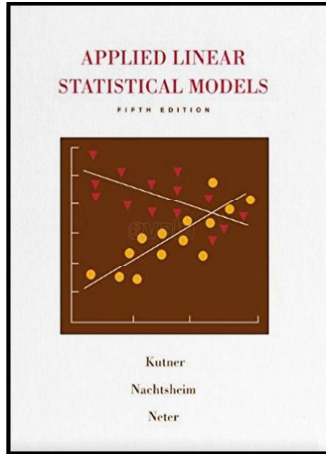


# Applied Linear Statistical Models, Part 1

## Section 2.1. Inferences Concerning $\beta_1$ —Proofs of Theorems



## Lemma 2.1.A

**Lemma 2.1.A.** Statistic  $b_1$  is a linear combination of the observations  $Y_i$ :

$$b_1 = \sum_{i=1}^n k_i Y_i \text{ where } k_i = \frac{X_i - \bar{X}}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

**Proof.** First, we have

$$\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y}) = \sum_{i=1}^n (X_i - \bar{X})Y_i - \sum_{i=1}^n (X_i - \bar{X})\bar{Y}.$$

But

$$\begin{aligned} \sum_{i=1}^n (X_i - \bar{X})\bar{Y} &= \bar{Y} \sum_{i=1}^n (X_i - \bar{X}) = \bar{Y} \sum_{i=1}^n \left( X_i - \sum_{i=1}^n X_i / n \right) \\ &= \bar{Y} \left( \sum_{i=1}^n X_i - \sum_{i=1}^n X_i \right) = 0. \end{aligned}$$

## Lemma 2.1.A (continued)

**Proof (continued).** So

$$\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y}) = \sum_{i=1}^n (X_i - \bar{X})Y_i - \sum_{i=1}^n (X_i - \bar{X})\bar{Y} = \sum_{i=1}^n (X_i - \bar{X})Y_i.$$

Then we have

$$b_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})Y_i}{\sum_{i=1}^n (X_i - \bar{X})^2} = \sum_{i=1}^n k_i Y_i,$$

where  $k_i = \frac{X_i - \bar{X}}{\sum_{i=1}^n (X_i - \bar{X})^2}$ , as claimed.  $\square$

## Theorem 1.11

**Theorem 1.11. The Gauss-Markov Theorem for the Normal Error Regression Model.**

Consider the data points  $(X_i, Y_i)$  for  $i = 1, 2, \dots, n$  and the normal error linear regression model  $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$  given in (1.1) with the added hypothesis that each error term has a  $N(0, \sigma^2)$  distribution. The least squares estimators

$$b_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2} \text{ and } b_0 = \frac{1}{n} \left( \sum_{i=1}^n Y_i - b_1 \sum_{i=1}^n X_i \right) = \bar{Y} - b_1 \bar{X}$$

are unbiased (that is,  $E\{b_0\} = \beta_0$  and  $E\{b_1\} = \beta_1$ ) and have minimum variance among all unbiased linear estimators (i.e., linear combinations of the  $Y_i$ ).

**Proof.** First we have by Lemma 2.1.A and the linearity of expectation (see my online notes for Mathematical Statistics 1 [STAT 4047/5047] on [Section 1.8. Expectation of a Random Variables](#); notice Theorem 1.8.2) we have: ...

## Theorem 1.11 (continued 1)

**Proof (continued).** ...

$$\begin{aligned} E\{b_1\} &= E\left\{\sum_{i=1}^n k_i Y_i\right\} = \sum_{i=1}^n k_i E\{Y_i\} \\ &= \sum_{i=1}^n k_i(\beta_0 + \beta_1 X_i) = \beta_0 \sum_{i=1}^n k_i + \beta_1 \sum_{i=1}^n k_i X_i = \beta_1, \end{aligned}$$

because  $\sum_{i=1}^n k_i = 0$  by (2.5) and  $\sum_{i=1}^n k_i X_i = 1$  by Exercise 2.50, as claimed.

Recall that for independent  $Y_i$ , we have  $\sigma^2\left(\sum_{i=1}^n a_i Y_i\right) = \sum_{i=1}^n a_i^2 \sigma^2(Y_i)$ ;

see equation A.31 in Appendix A or my online notes on Mathematical Statistics 1 on [Section 1.8. Expectation of a Random Variables](#) (see Theorem 1.8.2) and on [Section 1.9. Some Special Expectations](#) (see Theorem 1.9.1). We use this to find  $\sigma\{b_1\}$ .

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## Theorem 1.11 (continued 2)

**Proof (continued).** We now have by Lemma 2.1.A and (2.7) of Note 2.1.A that

$$\begin{aligned} \sigma^2\{b_1\} &= \sigma^2\left\{\sum_{i=1}^n k_i Y_i\right\} = \sum_{i=1}^n k_i^2 \sigma^2\{Y_i\} \\ &= \sum_{i=1}^n k_i^2 \sigma^2 = \sigma^2 \frac{1}{\sum_{i=1}^n (X_i - \bar{X})^2}. \end{aligned}$$

Next, we need to show that among all unbiased linear estimators of  $\beta_1$  of the form  $\hat{\beta}_1 = \sum_{i=1}^n c_i Y_i$ , the one of minimum variance is the one for which  $c_i = k_i$  for  $i = 1, 2, \dots, n$ . Since  $\hat{\beta}_1$  must be unbiased, then

$$E\{\hat{\beta}_1\} = E\left\{\sum_{i=1}^n c_i Y_i\right\} = \sum_{i=1}^n c_i E\{Y_i\} = \beta_1.$$

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## Theorem 1.11 (continued 3)

**Proof (continued).** We know  $E\{Y_i\} = \beta_0 + \beta_1 X_i$  by Note 1.3.A, so we now have

$$\begin{aligned} E\{\hat{\beta}_1\} &= \sum_{i=1}^n c_i E\{Y_i\} = \sum_{i=1}^n c_i(\beta_0 + \beta_1 X_i) \\ &= \beta_0 \sum_{i=1}^n c_i + \beta_1 \sum_{i=1}^n c_i X_i = \beta_1. \end{aligned}$$

For this to hold for arbitrary normal error linear regression (and hence to hold for all  $\beta_0$  and  $\beta_1$ ) we must have  $\sum_{i=1}^n c_i = 0$  and  $\sum_{i=1}^n c_i X_i = 1$ .

As described above, the variance of  $\hat{\beta}_1$  satisfies:

$$\sigma^2\{\hat{\beta}_1\} = \sigma^2\left\{\sum_{i=1}^n c_i Y_i\right\} = \sum_{i=1}^n c_i^2 \sigma^2\{Y_i\} = \sigma^2 \sum_{i=1}^n c_i^2.$$

Define  $d_i$  so that it satisfies  $c_i = k_i + d_i$ , where the  $k_i$  are defined in Lemma 2.1.A.

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## Theorem 1.11 (continued 4)

**Proof (continued).** The variance of  $\hat{\beta}_1$  then becomes

$$\sigma^2\{\hat{\beta}_1\} = \sigma^2 \sum_{i=1}^n c_i^2 = \sigma^2 \sum_{i=1}^n (k_i + d_i)^2 = \sigma^2 \left( \sum_{i=1}^n k_i^2 + \sum_{i=1}^n d_i^2 + 2 \sum_{i=1}^n k_i d_i \right).$$

$$\begin{aligned} \text{Now } \sum_{i=1}^n k_i d_i &= \sum_{i=1}^n k_i (c_i - k_i) = \sum_{i=1}^n c_i k_i - \sum_{i=1}^n k_i^2 \\ &= \sum_{i=1}^n c_i \left( \frac{X_i - \bar{X}}{\sum_{i=1}^n (X_i - \bar{X})^2} \right) - \frac{1}{\sum_{i=1}^n (X_i - \bar{X})^2} \\ &\text{by Lemma 2.1.A and (2.7) of Note 2.1.A} \\ &= \frac{\sum_{i=1}^n c_i X_i - \bar{X} \sum_{i=1}^n c_i}{\sum_{i=1}^n (X_i - \bar{X})^2} - \frac{1}{\sum_{i=1}^n (X_i - \bar{X})^2} = 0 \end{aligned}$$

since  $\sum_{i=1}^n c_i = 0$  and  $\sum_{i=1}^n c_i X_i = 1$ , as shown above.

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## Theorem 1.11 (continued 5)

**Proof (continued).** So the variance of  $\hat{\beta}_1$  becomes

$$\sigma^2\{\hat{\beta}_1\} = \sigma^2 \left( \sum_{i=1}^n k_i^2 + \sum_{i=1}^n d_i^2 \right).$$

We have  $\sigma^2\{b_1\} = \sigma^2 \sum_{i=1}^n k_i^2$  as shown above. Therefore  $\sigma^2\{\hat{\beta}_1\} = \sigma^2\{b_1\} + \sum_{i=1}^n d_i^2$ . So the variance of  $\hat{\beta}_1$  is minimized with all  $d_i = 0$ , and hence when  $c_i = k_i$  for all  $i = 1, 2, \dots, n$ . That is, the variance of  $\hat{\beta}_1$  is minimized when  $\hat{\beta}_1 = b_1$ , as claimed.

In Exercise 2.51 it is to be shown that  $b_0$  is an unbiased estimator of  $\beta_0$  (as claimed):  $E\{b_0\} = \beta_0$ .