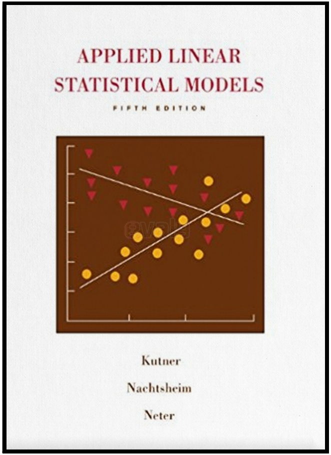
Applied Linear Statistical Models, Part 1

Section 5.8. Random Vectors and Matrices—Proofs of Theorems



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Theorem 5.8.A

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(5.44) Expectation of a Constant Matrix: $\mathbf{E}\{\mathbf{A}\} = \mathbf{A}$.

(5.45) Expectation of a Constant Matrix Times a Random Vector:

$$\mathbf{E}\{\mathbf{W}\} = \mathbf{E}\{\mathbf{AY}\} = \mathbf{AE}\{\mathbf{Y}\}.$$

(5.46) Variance of a Constant Matrix Times a Random Vector:

$$\boldsymbol{\sigma}^2\{\mathbf{W}\} = \boldsymbol{\sigma}^2\{\mathbf{AY}\} = \mathbf{A}\boldsymbol{\sigma}^2\{\mathbf{Y}\}\mathbf{A}'.$$

Proof. (5.44) This follows from the definition of the expectation of a matrix and the fact that the expectation of a constant is the constant itself, as claimed.

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Theorem 5.8.A (continued 1)

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(5.45) Expectation of a Constant Matrix Times a Random Vector:

$$\mathbf{E}\{\mathbf{W}\} = \mathbf{E}\{\mathbf{AY}\} = \mathbf{AE}\{\mathbf{Y}\}.$$

Proof (continued). (5.45) Let $A = [a_{ij}]$ be an $m \times n$ matrix of scalars and let $\mathbf{Y} = [Y_j]$ be an $n \times 1$ vector of random variables. Then the ith entry of $\mathbf{AY}$ is $\sum_{j=1}^n a_{ij} Y_j$. So the ith entry of $\mathbf{E}\{\mathbf{AY}\}$ is $\mathbf{E}\left\{\sum_{j=1}^n a_{ij} Y_j\right\} = \sum_{j=1}^n a_{ij} E\{Y_j\}$, because by Theorem 1.8.2 in my online notes for Mathematical Statistics 1 [STAT 4047/5047] on Section 1.8. Expectation of a Random Variable, expectation is linear. Similarly, the ith entry of $\mathbf{AE}\{\mathbf{Y}\}$ is $\sum_{j=1}^n a_{ij} E\{Y_j\}$ and so $\mathbf{E}\{\mathbf{AY}\} = \mathbf{AE}\{\mathbf{Y}\}$, as claimed.

Theorem 5.8.A (continued 2)

Theorem 5.8.A. For **Y** a random vector and **A** a matrix of scalars (i.e., constants), we can define the new random variable $\mathbf{W} = \mathbf{AY}$. Then:

(5.46) Variance of a Constant Matrix Times a Random Vector:

$$\boldsymbol{\sigma}^2\{\mathbf{W}\} = \boldsymbol{\sigma}^2\{\mathbf{AY}\} = \mathbf{A}\boldsymbol{\sigma}^2\{\mathbf{Y}\}\mathbf{A}'.$$

Proof (continued). (5.46) With notation introduced for (5.45), the (i,j)-entry of $\boldsymbol{\sigma}^2\{\mathbf{AY}\}$ is $\mathbf{E}\left\{\left(\sum_{j=1}^n a_{ij}E\{Y_j\} - E\left(\sum_{j=1}^n a_{ij}E\{Y_j\}\right)\right)^2\right\}$. By Note 5.8.A,

$$\boldsymbol{\sigma}^2\{\mathbf{W}\} = \mathbf{E}\left\{[W_j - E\{W_j\}][W_j - E\{W_j\}]'\right\}$$

$$= \mathbf{E}\left\{\left[\sum_{j=1}^n a_{ij}Y_j - E\left\{\sum_{j=1}^n a_{ij}Y_j\right\}\right]\left[\sum_{j=1}^n a_{ij}Y_j - E\left\{\sum_{j=1}^n a_{ij}Y_j\right\}\right]'\right\}$$

Theorem 5.8.A (continued 3)

Proof (continued). ...

$$
\begin{aligned}
\boldsymbol{\sigma}^2\{\mathbf{W}\} &= \mathbf{E}\left\{\left[\sum_{j=1}^{n} a_{ij}Y_j - \sum_{j=1}^{n} a_{ij}E\{Y_j\}\right]\left[\sum_{j=1}^{n} a_{ij}Y_j - \sum_{j=1}^{n} a_{ij}E\{Y_j\}\right]'\right\} \\
&= \mathbf{E}\{(\mathbf{A}[Y_j - E\{Y_j\}])(\mathbf{A}[Y_j - E\{Y_j\}]'\} \\
&= \mathbf{E}\{\mathbf{A}[Y_j - E\{Y_j\}][Y_j - E\{Y_j\}]'\mathbf{A}'\} \text{ by (5.32)} \\
&= \mathbf{A}\mathbf{E}\{[Y_j - E\{Y_j\}][Y_j - E\{Y_j\}]'\}\mathbf{A}' \text{ since } \mathbf{E} \text{ is linear} \\
&= \mathbf{A}\boldsymbol{\sigma}^2\{\mathbf{Y}\}\mathbf{A}' \text{ by Note 5.8.A,}
\end{aligned}
$$

as claimed. □