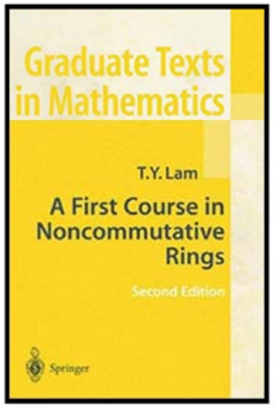
Noncommutative Ring Theory

Chapter 1. Wedderburn-Artin Theory

1.1. Basic Terminology and Examples—Proofs of Theorems



Lemma 1.1.A

Lemma 1.1.A. Ring R is reduced if and only if for all $a \in R$ with $a^2 = 0$ we have $a = 0$.

Proof. If R is reduced then for $a \in R$ a nonzero element, we have $a^n \neq 0$ for all $n \in \mathbb{N}$ and, in particular, $a^2 \neq 0$. So the only $a \in R$ where $a^2 = 0$ is $a = 0$, as claimed.

Suppose that for every $a \in R$ such that $a^2 = 0$ we have $a = 0$. ASSUME $b \in R$ is a nonzero nilpotent element. Then there is $n \in \mathbb{N} \setminus \{1\}$ such $b^n = 0$ and $b^{n-1} \neq 0$ (that is, n is the smallest positive value such that $b^n = 0$). Notice that $n > 2$, or else y contradicts the hypothesis for this case. Let $a = b^{n-1}$. Then $a \neq 0$ and $a^2 = (b^{n-1})^2 = b^{2n-2} = b^n b^{n-2} = 0b^{n-2} = 0$ (notice that $n > 2$), a CONTRADICTION to the hypothesis for this case. So the assumption that there exists a nonzero nilpotent element in R is false, and therefore R has no nonzero nilpotent elements. That is, R is reduced. □