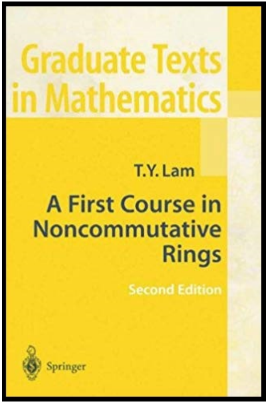
Noncommutative Ring Theory

Chapter 1. Wedderburn-Artin Theory

1.1. Basic Terminology and Examples—Proofs of Theorems



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Suppose that for every $a \in R$ such that $a^2 = 0$ we have $a = 0$. ASSUME $b \in R$ is a nonzero nilpotent element. Then there is $n \in \mathbb{N} \setminus \{1\}$ such $b^n = 0$ and $b^{n-1} \neq 0$ (that is, n is the smallest positive value such that $b^n = 0$). Notice that $n > 2$, or else y contradicts the hypothesis for this case. Let $a = b^{n-1}$.

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