Chapter 1. Wedderburn-Artin Theory

Note. Lam states on page 1: "Modern ring theory began when J.J.M. Wedderburn proved his celebrated classification theorem for finite dimensional semisimple algebras over a field. . . . The Wedderburn-Artin theory has since become the cornerstone of noncommutative ring theory. . . " The Wedderburn-Artin Theorem appears as Theorem IX.1.14 in Hungerford. Lam declares this result as ". . . one of the earliest (and still one of the nicest) complete classification theorems in abstract algebra. . . ."

Section 1.1 Basic Terminology and Examples

Note. Throughout Lam's book, **all rings are assumed to be rings with unity (i.e., multiplicative identities)**, though not necessarily commutative.

Note. We quickly recall a couple of basic definitions from Hungerford:

Definition III.1.1. A *ring* is a nonempty set R together with two binary operations (denoted $+$ and multiplication) such that:

(i) $(R, +)$ is an abelian group.

(ii) $(ab)c = a(bc)$ for all $a, b, c \in R$ (i.e., multiplication is associative).

(iii) $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ (left and right distribution of multiplication over $+$).

If in addition,

(iv) $ab = ba$ for all $a, b \in R$,

then R is a *commutative ring*. If R contains an element 1_R such that

(v) $1_R a = a 1_R = a$ for all $a \in R$,

then R is a *ring with identity* (or *unity*).

Definition III.2.1. Let R be a ring and S a nonempty subset of R that is closed under the operations of addition and multiplication in R. If S is itself a ring under these operations then S is a *subring* of R. A subring I of R is a *left ideal* provided

$$r \in R \text{ and } x \in I \text{ implies } rx \in I;$$

I is a *right ideal* provided

$$r \in R \text{ and } x \in I \text{ implies } xr \in I;$$

I is an *ideal* if it is both a left and right ideal.

Definition. A *simple ring* is a ring R whose only ideals are (0) and R.

Definition. A *right zero divisor* of a is an element b such that $ab = 0$. *Left zero divisor* is defined similarly. A *zero divisor* is an element which is both a left and right zero divisor.

Example of one-sided zero divisor. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. That is R is the ring of matrices of the form $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ where $x, z \in \mathbb{Z}$ and $y \in \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. Let $a = \begin{bmatrix} 2 & \overline{0} \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & \overline{1} \\ 0 & 0 \end{bmatrix}$. Then

$$ab = \begin{bmatrix} 2 & \overline{0} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \overline{1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2(0) + (\overline{0})0 & 2(\overline{1}) + (\overline{0})0 \\ 0(0) + 1(0) & 0(\overline{1}) + 1(0) \end{bmatrix} = \begin{bmatrix} 0 & \overline{0} \\ 0 & 0 \end{bmatrix}$$

(here multiplication means repeated addition, NOT some group multiplication). But $ba = \begin{bmatrix} 0 & \overline{1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & \overline{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \overline{1} \\ 0 & 0 \end{bmatrix}$. Thus b is a right zero divisor of a, but b is not a left zero divisor of a.

Definition. A *domain* is a ring with no left or right zero divisors.

Definition. A nonzero element a of a ring is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.

Definition. A ring R is *reduced* if R has no nonzero nilpotent elements.

Lemma 1.1.A. Ring R is reduced if and only if for all $a \in R$ with $a^2 = 0$ we have $a = 0$.

Definition. Let R be a ring (with unity). An element $a \in R$ is *right-invertible* if there exists $b \in R$ such that $ab = 1$. Element $a \in R$ is *left-invertible* if there exists $b' \in R$ such that $b'a = 1$. If a has both left and right inverses b' and b, respectively, (in which case $b' = b'a = b'(ab) = (b'a)b = 1b = b$), then a is *invertible* (or a is a *unit*) and $b = b'$ is the *inverse* of a, denoted a^{-1}.

Note. We denote the set of all units of a ring R (with unity) as $U(R)$. In fact, $U(R)$ is a group with the multiplication of R as the binary operation (this is a standard result; see Exercise IV.18.37 in Fraleigh's *A First Course in Abstract Algebra*, 7th edition).

Example. Consider the ring of infinite matrices with real entries such that each row and each column contains only finitely many nonzero entries. We use the usual definition for matrix addition and multiplication (and the finitely many nonzero entries insures convergence of matrix products). Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then $BA = \mathcal{I}$ but

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \neq \mathcal{I}.$$

So B is a left inverse of A but not a right inverse of A; A is left invertible and B is right invertible. We can get a geometric interpretation of this by applying A to an infinite column vector $[x_1, x_2, x_3, \ldots]^T$. This yields $[0, x_1, x_2, \ldots]^T$, so A corresponds to a "right shift operator" when applied to sequences. Applying B to the same column vector gives $[x_2, x_3, x_4, \ldots]^T$, so B corresponds to a "left shift operator." So it makes sense that A followed by B (represented by the product BA) gives an identity whereas B followed by A (represented by the product AB) results in changing entry x_1 to 0.

Definition. A ring in which the right-invertibility of an element implies the left invertibility of that element (that is, $ab = 1$ implies $ba = 1$) is *Dedekind-finite* (or *von Neumann-finite*).

Note. The motivation for the "finite" term in the previous definition is suggested by the previous example involving infinite matrices (which Lam gives on page 4 with a slightly different notation) and the fact that in the ring of $n \times n$ matrices $\mathrm{Mat}_n(D)$, where D is a division ring, right-invertibility implies left invertibility (by Hungerford's Exercise VII.2.6; see also Theorem 1.11, "A Commutative Property," of Fraleigh and Beauregard's *Linear Algebra*, 3rd edition or my online notes for Section 1.5 "Inverses of Square Matrices" of this work at `http://faculty.etsu.edu/gardnerr/2010/c1s5.pdf` where this property is shown for $n \times n$ real matrices). If V is a vector space of dimension n over division ring D then the ring of endomorphisms of V (an endomorphism of V is a homomorphism mapping $V \to v$), then $\mathrm{Hom}_D(V,V) = \mathrm{End}_D(V) \cong \mathrm{Mat}_n(D')$ (by Hungerford's Theorem VII.1.4, the "in particular" part as stated below). So $\mathrm{End}_D(V)$ is Dedekind-finite whenever V is finite dimensional.

Note. Hungerford's Theorem VII.1.4 in its entirety states (with a mixture of Hungerford's and Lam's notation here) that:

> **Theorem VII.1.4.** Let R be a ring with identity and E a free left R-module with a finite basis of n elements. Then there is an isomorphism of rings:
>
> $$\mathrm{Hom}_R(E,E) = \mathrm{End}_R(E) \cong \mathrm{Mat}_n(R^{\mathrm{op}}).$$
>
> In particular, this isomorphism exists for every n-dimensional vector space E over a division ring R, in which case R^{op} is also a division ring.

The ring R^{op} is defined as follows.

Definition. Let R be a ring. Define ring R^{op} as the image of a bijection mapping $R \to R^{\mathrm{op}}$ where we denote the image of $a \in R$ as $a^{\mathrm{op}} \in R^{\mathrm{op}}$ and

1. $a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}}$, and

2. $a^{\mathrm{op}} b^{\mathrm{op}} = (ba)^{\mathrm{op}}$.

Note. In Hungerford's Exercise II.1.17(a), it is to be shown that R^{op} is in fact a ring. Also, R has an identity if and only if R^{op} does (part (b) of the exercise), and R is a division ring if and only if R^{op} is (part (c)). We see that R^{op} arises naturally as given in Hungerford's Theorem VII.1.4.

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