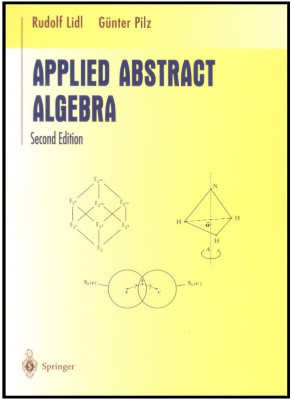
Applied Abstract Algebra

Chapter 7. Further Applications of Algebra

7.31. Semigroups and Biology—Proofs of Theorems



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Proof. Recall that, by the definition of sequences, we have $a_1 a_2 \cdots a_n = a'_1 a'_2 \cdots a'_m$ if and only if $n = m$ and $a_i = a'_i$ for all $i \in \{1, 2, \ldots, n\}$. So every element of the free group A_* is a unique product of elements of A; hence A is a generating set of A_* Often called a *basis* of A_*). So every map from A to some semigroup S can be uniquely extended to a homomorphism from A_* to S by simply defining the image of a product of generators as the product of the images. For example, $f(a_1 a_2 \cdots a_n) = f(a_1) f(a_2) \cdots f(a_n)$.

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Theorem 7.31.2 (continued)

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Proof (continued). Consider the injection g on $\{a_1, a_2, \ldots, a_{21}\}$ defined as:

$$\begin{array}{llll} g(a_1) = n_1 n_1 n_1, & g(a_2) = n_1 n_1 n_2, & g(a_3) = n_1 n_1 n_3, & g(a_4) = n_1 n_1 n_4, \\ g(a_5) = n_1 n_2 n_1, & g(a_6) = n_1 n_2 n_2, & g(a_7) = n_1 n_2 n_3, & g(a_8) = n_1 n_2 n_4, \\ g(a_9) = n_1 n_3 n_1, & g(a_{10}) = n_1 n_3 n_2, & g(a_{11}) = n_1 n_3 n_3, & g(a_{12}) = n_1 n_3 n_4, \\ g(a_{13}) = n_1 n_4 n_1, & g(a_{14}) = n_1 n_4 n_2, & g(a_{15}) = n_1 n_4 n_3, & g(a_{16}) = n_1 n_4 n_4, \\ g(a_{17}) = n_2 n_1 n_1, & g(a_{18}) = n_2 n_1 n_2, & g(a_{19}) = n_2 n_1 n_3, & g(a_{20}) = n_2 n_1 n_4, \\ g(a_{21}) = n_2 n_2 n_1. & & & \end{array}$$

Then g is injective and so extends to a monomorphism on F_{21}. Notice that g maps each a_i to a word in F_4 of length 3. For each $n \geq 4$, by using words of lengths n as the images of the a_i's we can similar define a different injective g which extends to a different monomorphism. Therefore there are infinitely many monomorphism from F_{21} into F_4, as claimed. $\square$

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