1.2. Basic Elements

Note. In this section, we state several definitions we need to get started. We consider three examples to illustrate the definitions.

Definition. The unknown quantity θ which affects the discussion process in the *state of nature*. The set of all possible state of nature is denoted Θ. When an experiment is performed to gather information about θ, θ is a *parameter* and Θ is the *parameter space*.

Definition. A *loss function* $L(\theta, a)$ is a real valued function with domain $\Theta \times \mathscr{A}$, where $\mathscr{A}$ is the set of all possible *actions* (that is, decisions), and $L(\theta, a) \geq -K > -\infty$ for all $(\theta, a) \in \Theta \times \mathscr{A}$ and for some $-K \in \mathbb{R}$.

Note. We consider how to determine a loss function in Chapter 2.

Note/Definition. Parameter θ is estimated by random variable X. When we experiments are performed to estimate θ, the outcomes of the experiments will be denoted $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ (we denote vectors with boldfaced fonts). A particular realization of random variable X is denoted x. The set of all possible outcomes of the experiments is the *sample space* of X, denoted $\mathscr{X}$. For event $A \subset \mathscr{A}$, we let $P_\theta(A) = P_\theta(X \in A)$ denote the probability of the event A when

θ is the true state of nature. We denote the probability density function or X when θ is the true state of nature as $f(x|\theta)$. If X is a continuous random variable, then $P_\theta(A) = \displaystyle\int_A f(x|\theta)\,dx$ and if X is a discrete random variable then $P_\theta(A) = \displaystyle\sum_{xinA} f(x|\theta)$. Notice that Berger mentions Lebesgue measure in connection with a continuous random variable. Lebesgue measure and integration are the topics of ETSU's Real Analysis 1 (MATH 5210); see my online notes for Real Analysis 1 for more details. In fact, a graduate class in probability would make extensive use of these ideas; see my online notes for Measure Theory Based Probability (not a formal ETSU class).

Definition. The *expectation* over X of function $h(x)$, for a given value of θ, is

$$E_\theta[h(X)] = \begin{cases} \int_{\mathscr{X}} j(x) f(x|\theta)\,dx & \text{(continuous case)},\\ \sum_{x\in\mathscr{X}} h(x) f(x|\theta) & \text{(discrete case)}.\end{cases}$$

In both cases the continuous and the discrete case we denote the expectation as $E_\theta[h(X)] = \displaystyle\int_{\mathscr{X}} h(x)\,dF^X(x|\theta)$.

Note. We may take $E_\theta[h(X)] = \displaystyle\int_{\mathscr{X}} h(x)\,dF^X(x|\theta)$ simply as notation for the computations above, or equivalently we may interpret it as a Riemann-Stieltjes integral where $F^X(x|\theta)$ is the cumulative distribution function of X. This allows us to encompass both continuous and discrete cases in one computation. Details on Riemann-Stieltjes integrals (and the use of the Dirac-Delta distribution) are in my online notes for Analysis 1 (MATH 4217/5217) on Section 6.2. Some Properties and Applications of the Riemann Integral. Information on the more general Lebesgue-Stieltjes integral is covered in my online notes for Measure Theory Based Probability (not a formal ETSU class) on Section 1.4. Lebesgue-Stieltjes Measure and Distribution Functions. Since $F^X(x|\theta)$ is the cumulative distribution function of X given θ, then we can also calculate the probability of event A as $P_\theta(A) = \int_A dF^X(x|\theta)$ in both the continuous and discrete cases.

Revised: 2/26/2022