Chapter 1. Basic Counting

Note. In this first chapter, we present the "most elementary techniques" for enumeration. The techniques will increase in difficulty in the following chapters. Notationally, we denote the natural numbers, integers, rationals, reals, and complex numbers as usual: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. In this book, the natural numbers are taken as $\mathbb{N} = \{1, 2, 3, \ldots\}$ (which is, surprisingly, not universal) and we denote the nonnegative integers as $\mathbb{P} = \{0, 1, 2, 3, \ldots\}$. Throughout this course, **whenever we refer to the cardinality of a set we will assume that it is finite!** Cardinalities of infinite sets can be exotic (as one sees in a set theory class), but we can come up with plenty of problems related to finite sets...

1.1. The Sum and Product Rules for Sets

Note. In this section, we start with the Sum Rule and the Product Rule concerning cardinalities of (finite) sets. We introduce a dominos-style tiling problem and use it to introduce the Fibonacci numbers. In so doing, we see our first combinatorial proof (see Corollary 1.1.3).

Note. You have no doubt seen the Sum Rule and Product Rule for cardinalities of sets. You would see it in Foundations of Probability and Statistics-Calculus (MATH 2050), Mathematical Reasoning (MATH 3000; see my online notes for this class on Section 4.1. Cardinality; Fundamental Counting Principles, see Theorems 4.14 and 4.17), or Applied Combinatorics and Problem Solving (MATH 3340). Before stating these results, we introduce some notation that may be new to you.

For finite set S, we denote its cardinality (that is, the number of elements in set S) as $|S|$ or $\$S$. If two sets S and T are disjoint, then we denote their disjoint union as $S \uplus T$; a more common notation for this (in the analysis world) is $S \mathbin{\dot\cup} T$.

Lemma 1.1.1. Let S and T be finite sets.

(a) Sum Rule. If $S \cap T = \varnothing$, then $|S \uplus T| = |S| + |T|$.

(b) Product Rule. For any finite sets, $|S \times T| = |S| \cdot |T|$.

Note. Recall that a bijection between sets S and T is a function $f : S \to T$ such that f is injective (or "one to one) and surjective (or onto). Recall that two sets have the same cardinality (by definition) if there is a bijection between the sets; the holds for both finite or infinite sets (though we are not concerned with infinite sets here). See my online notes for Mathematical Reasoning (MATH 3000) on Section 4.1. Cardinality; Fundamental Counting Principles (notice Definition 4.1).

Note. Fibonacci (circa 1170–circa 1245), or Leonardo of Pisa, is best know for authoring *Liber Abaci* ("Book of Calculation") in 1202 that resulted in the widespread use of the Hindu-Arabic number system throughout Europe. In this book he mentions what is now known as the Fibonacci sequence, though the sequence had been known in Indian mathematics well before Fibonacci. He described it in terms of the number of rabbits present in an environment, given a founding pair (with an oversimplification of the actual biology). This is described in more detail in my online notes for Linear Algebra (MATH 2010) on Section 5.1. Eigenvalues and

Eigenvectors (notice the first example, which introduces the Fibonacci sequence). The sequence is defined by the recursive equation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with the initial values $F_0 = 0$ and $F_1 = 1$. The terms F_n are the *Fibonacci numbers*. A general formula for F_n can be derived by diagonalizing a matrix and raising it to the nth power, as is done in Linear Algebra; see Section 5.3. Two Applications (see "Page 318 Example 2" which gives a formula for the nth Fibonacci number). You may have also encountered the Fibonacci sequence in Mathematical Reasoning (MATH 3000); see my online notes for Mathematical Reasoning on Section 2.10. Mathematical Induction and Recursion, which briefly introduces the Fibonacci sequence in the setting of recursion. In this class, we give a formula for F_n in Section 3.6. Recurrence Relations and Generating Functions.

Note. We next give a "combinatorial interpretation" of the Fibonacci numbers. That is, we seek a sequence of sets $S_0, S_1, S_2, \ldots$ such that $\#S_n = F_n$ for all n (other than the artificial rabbit story given in *Liber Abaci*). Consider a row of squares. We have access to two types of tiles, dominos which can cover two squares and monominos which cover one square. A *tiling* of the row is a set of tiles which covers each square exactly once. Let $\mathscr{T}_n$ be the set of tilings of a row of n squares. Figure 1.1 gives $\mathscr{T}_3$. The next result relates the Fibonacci numbers and $\#\mathscr{T}_n$.

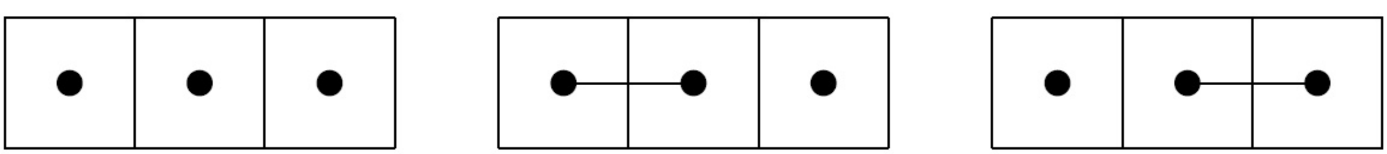

Figure 1.1. The 3 tilings of $\mathscr{T}_3$.

Theorem 1.1.2. For $n \geq 1$ we have $F_n = \#\mathscr{T}_{n-1}$.

Note. We can use this relationship between the Fibonacci numbers and $\#\mathscr{T}$ to give an identity concerning the Fibonacci numbers.

Corollary 1.1.3. For $m \geq 1$ and $n \geq 0$ we have $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$.

Note. The technique of proof given for Corollary 1.1.3 is a "combinatorial proof" since it involves counting discrete objects. Sagan state (see page 4): "... combinatorial proofs are often considered to be the most pleasant, in part because they can be more illuminating than demonstrations just involving formal manipulations."

Revised: 2/15/2022