Graph Theory

Chapter 1. Ranking

1.2. Results—Proofs of Theorems

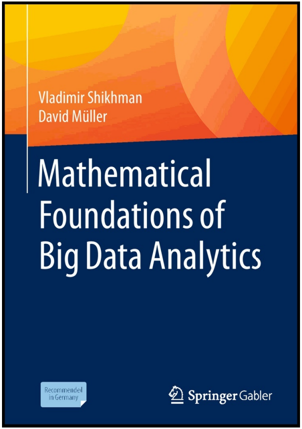

Table of contents

Lemma 1.2.A

Lemma 1.2.A. If P is an $n \times n$ stochastic matrix, then $\det(P - I) = 0$ where I is the $n \times n$ identity matrix.

Proof. Since $P = [p_{ij}]$ is a stochastic matrix, then the sum of the entries of each column is 1, $\sum_{i=1}^{n} p_{ij} = 1$ for each $j = 1, 2, \ldots, n$. Recall that the determinant is unaffected by row addition (see The Row-Addition Property of Theorem 4.2.A. Properties of the Determinant in my online Linear Algebra [MATH 2010] notes on 4.2. The Determinant of a Square Matrix).

Lemma 1.2.A

Lemma 1.2.A. If P is an $n \times n$ stochastic matrix, then $\det(P - I) = 0$ where I is the $n \times n$ identity matrix.

Proof. Since $P = [p_{ij}]$ is a stochastic matrix, then the sum of the entries of each column is 1, $\sum_{i=1}^{n} p_{ij} = 1$ for each $j = 1, 2, \ldots, n$. Recall that the determinant is unaffected by row addition (see The Row-Addition Property of Theorem 4.2.A. Properties of the Determinant in my online Linear Algebra [MATH 2010] notes on 4.2. The Determinant of a Square Matrix). So we have

$$\det(P - I) = \det\left(\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\right)$$

Lemma 1.2.A

Lemma 1.2.A. If P is an $n \times n$ stochastic matrix, then $\det(P - I) = 0$ where I is the $n \times n$ identity matrix.

Proof. Since $P = [p_{ij}]$ is a stochastic matrix, then the sum of the entries of each column is 1, $\sum_{i=1}^{n} p_{ij} = 1$ for each $j = 1, 2, \ldots, n$. Recall that the determinant is unaffected by row addition (see The Row-Addition Property of Theorem 4.2.A. Properties of the Determinant in my online Linear Algebra [MATH 2010] notes on 4.2. The Determinant of a Square Matrix). So we have

$$\det(P - I) = \det\left(\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\right)$$

Lemma 1.2.A (continued 1)

Proof (continued).

$$
\begin{aligned}
\det(P - I) &= \det \begin{pmatrix} p_{11}-1 & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22}-1 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-1 \end{pmatrix} \\
&= \det \begin{pmatrix} \sum_{i=1}^n p_{i1} - 1 & \sum_{i=1}^n p_{i2} - 1 & \cdots & \sum_{i=1}^n p_{in} - 1 \\ p_{21} & p_{22}-1 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-1 \end{pmatrix}
\end{aligned}
$$

by adding Rows 2 through n to Row 1

Lemma 1.2.A (continued 2)

Proof (continued).

$$\det(P-I) = \det\begin{pmatrix} \sum_{i=1}^n p_{i1}-1 & \sum_{i=1}^n p_{i2}-1 & \cdots & \sum_{i=1}^n p_{in}-1 \\ p_{21} & p_{22}-1 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & 0 & \cdots & 0 \\ p_{21} & p_{22}-1 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn}-1 \end{pmatrix}$$

since $\sum_{i=1}^n p_{ij} = 1$ for each j

$= 0$ expanding the determinant along the first row.

So $\det(P-I) = 0$, as claimed. □

Lemma 1.2.B

Lemma 1.2.B. For P is an $n \times n$ stochastic matrix, if $(\mathcal{Z})$ is feasible:

$$z \geq Pz,\ z \geq 0,\ e^T z \geq 1, \qquad (\mathcal{Z})$$

then $(\mathcal{X})$ is feasible:

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

Proof. Suppose $(\mathcal{Z})$ is feasible and let $z \in \mathbb{R}^n$ satisfy $(\mathcal{Z})$. Consider $x = \dfrac{z}{e^T z}$. We then have by the linearity of matrix multiplication that

$$x - Px = \frac{z}{e^T z} - P\left(\frac{z}{e^T z}\right) = \frac{z}{e^T z} - \frac{Pz}{e^T z} = \frac{1}{e^T z}(z - Pz) \geq 0$$

where the last inequality holds because $z \geq Pz$ and so $z - Pz \geq 0$.

Lemma 1.2.B

Lemma 1.2.B. For P is an $n \times n$ stochastic matrix, if $(\mathcal{Z})$ is feasible:

$$z \geq Pz,\ z \geq 0,\ e^T z \geq 1, \qquad (\mathcal{Z})$$

then $(\mathcal{X})$ is feasible:

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

Proof. Suppose $(\mathcal{Z})$ is feasible and let $z \in \mathbb{R}^n$ satisfy $(\mathcal{Z})$. Consider $x = \dfrac{z}{e^T z}$. We then have by the linearity of matrix multiplication that

$$x - Px = \frac{z}{e^T z} - P\left(\frac{z}{e^T z}\right) = \frac{z}{e^T z} - \frac{Pz}{e^T z} = \frac{1}{e^T z}(z - Pz) \geq 0$$

where the last inequality holds because $z \geq Pz$ and so $z - Pz \geq 0$. So the components of $x - Px$ are nonnegative and since the row vector $e^T P \in \mathbb{R}^n$ has as its jth component the sum of the entries in the jth column of P (which is 1), so $e^T P = e^T$.

Lemma 1.2.B

Lemma 1.2.B. For P is an $n \times n$ stochastic matrix, if $(\mathcal{Z})$ is feasible:

$$z \geq Pz,\ z \geq 0,\ e^T z \geq 1, \qquad (\mathcal{Z})$$

then $(\mathcal{X})$ is feasible:

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

Proof. Suppose $(\mathcal{Z})$ is feasible and let $z \in \mathbb{R}^n$ satisfy $(\mathcal{Z})$. Consider $x = \frac{z}{e^T z}$. We then have by the linearity of matrix multiplication that

$$x - Px = \frac{z}{e^T z} - P\left(\frac{z}{e^T z}\right) = \frac{z}{e^T z} - \frac{Pz}{e^T z} = \frac{1}{e^T z}(z - Pz) \geq 0$$

where the last inequality holds because $z \geq Pz$ and so $z - Pz \geq 0$. So the components of $x - Px$ are nonnegative and since the row vector $e^T P \in \mathbb{R}^n$ has as its jth component the sum of the entries in the jth column of P (which is 1), so $e^T P = e^T$.

Lemma 1.2.B (continued)

Proof (continued). So by associativity,

$$e^T(x - Px) = e^T x - e^T(Px) = e^T x - (e^T P)x = e^T x - e^T x = 0,$$

and hence the sum of the entries of nonnegative vector $x - Px$ is 0. That is $x - Px = 0$ and $x = Px$, so that the first condition of $(\mathcal{X})$ is satisfied. Next,

$$x = \frac{z}{e^T z} \geq 0 \text{ since } z \geq 0, \text{ and } e^T x = e^T \left(\frac{z}{e^T z}\right) = \frac{e^T z}{e^T z} = 1.$$

Hence, the other two conditions of $(\mathcal{X})$ are satisfied and so $(\mathcal{X})$ is also feasible as claimed, where we have $x = \dfrac{z}{e^T z}$. □

Lemma 1.2.B (continued)

Proof (continued). So by associativity,

$$e^T(x - Px) = e^T x - e^T(Px) = e^T x - (e^T P)x = e^T x - e^T x = 0,$$

and hence the sum of the entries of nonnegative vector $x - Px$ is 0. That is $x - Px = 0$ and $x = Px$, so that the first condition of $(\mathcal{X})$ is satisfied. Next,

$$x = \frac{z}{e^T z} \geq 0 \text{ since } z \geq 0, \text{ and } e^T x = e^T \left(\frac{z}{e^T z}\right) = \frac{e^T z}{e^T z} = 1.$$

Hence, the other two conditions of $(\mathcal{X})$ are satisfied and so $(\mathcal{X})$ is also feasible as claimed, where we have $x = \dfrac{z}{e^T z}$. □

Theorem 1.2.A

Theorem 1.2.A. The Weak Duality Theorem.
If $u \in \mathbb{R}^k$ is feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ is feasible for $(\mathcal{D})$, then we have

$$\min_{u \in \mathbb{R}^k} \{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v \in \mathbb{R}^m} \{b^T v \mid A^T v \leq c,\ v \geq 0\}.$$

Proof. For $u \in \mathbb{R}^k$ feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ feasible for $(\mathcal{D})$, we have

$$\begin{aligned} c^T u &\geq (A^T v)^T u \text{ since } c \geq A^T v \text{ by } (\mathcal{D}) \\ &= (v^T A)u = v^T (A^T)^T u = v^T (Au) \text{ by properties of transpose} \\ &\qquad \text{and associativity} \\ &\geq v^T b \text{ since } b \geq Au \text{ by } (\mathcal{P}) \\ &= b^T v \text{ since } v^T b = b^T v \text{ is a constant (a dot product).} \end{aligned}$$

Theorem 1.2.A

Theorem 1.2.A. The Weak Duality Theorem.
If $u \in \mathbb{R}^k$ is feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ is feasible for $(\mathcal{D})$, then we have

$$\min_{u \in \mathbb{R}^k} \{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v \in \mathbb{R}^m} \{b^T v \mid A^T v \leq c,\ v \geq 0\}.$$

Proof. For $u \in \mathbb{R}^k$ feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ feasible for $(\mathcal{D})$, we have

$$\begin{aligned} c^T u &\geq (A^T v)^T u \text{ since } c \geq A^T v \text{ by } (\mathcal{D}) \\ &= (v^T A)u = v^T (A^T)^T u = v^T (Au) \text{ by properties of transpose} \\ &\quad \text{and associativity} \\ &\geq v^T b \text{ since } b \geq Au \text{ by } (\mathcal{P}) \\ &= b^T v \text{ since } v^T b = b^T v \text{ is a constant (a dot product).} \end{aligned}$$

This inequality implies

$$\min_{u \in \mathbb{R}^k} \{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v \in \mathbb{R}^m} \{b^T v \mid A^T v \leq c,\ v \geq 0\},$$

as claimed. □

Theorem 1.2.A

Theorem 1.2.A. The Weak Duality Theorem.
If $u \in \mathbb{R}^k$ is feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ is feasible for $(\mathcal{D})$, then we have

$$\min_{u \in \mathbb{R}^k} \{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v \in \mathbb{R}^m} \{b^T v \mid A^T v \leq c,\ v \geq 0\}.$$

Proof. For $u \in \mathbb{R}^k$ feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ feasible for $(\mathcal{D})$, we have

$$\begin{aligned} c^T u &\geq (A^T v)^T u \text{ since } c \geq A^T v \text{ by } (\mathcal{D}) \\ &= (v^T A)u = v^T (A^T)^T u = v^T (Au) \text{ by properties of transpose} \\ &\quad \text{and associativity} \\ &\geq v^T b \text{ since } b \geq Au \text{ by } (\mathcal{P}) \\ &= b^T v \text{ since } v^T b = b^T v \text{ is a constant (a dot product).} \end{aligned}$$

This inequality implies

$$\min_{u \in \mathbb{R}^k} \{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v \in \mathbb{R}^m} \{b^T v \mid A^T v \leq c,\ v \geq 0\},$$

as claimed. □

Theorem 1.2.C

Theorem 1.2.C. For stochastic matrix P, the system

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

is feasible. That is, there exists $x \in \mathbb{R}^n$ satisfying the conditions of $(\mathcal{X})$.

Proof. First, we claim that $\overline{y} = e$ and $\overline{y}_{n+1} = 0$ give a solution to $(\mathcal{D}_{\mathcal{Z}})$. We have

$$\max_{y \in \mathbb{R}^n,\ y_{n+1} \in \mathbb{R}} \left\{ y_{n+1} \mid y \leq P^T y - y_{n+1} e,\ y \geq 0,\ y_{n+1} \geq 0 \right\}, \qquad (\mathcal{D}_{\mathcal{Z}})$$

and notice that $e = \overline{y} \leq P^T \overline{y} - \overline{y}_{n+1} e = P^T e - 0e = P^T e = e$ since P is a stochastic matrix and hence the sum of the entries of each column of P is 1 (and therefore the sum of the entries of each row of P^T is 1). Next we need to show the maximum of the admissible y_{n+1}'s does not exceed $\overline{y}_{n+1} = 0$.

Theorem 1.2.C

Theorem 1.2.C. For stochastic matrix P, the system

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

is feasible. That is, there exists $x \in \mathbb{R}^n$ satisfying the conditions of $(\mathcal{X})$.

Proof. First, we claim that $\overline{y} = e$ and $\overline{y}_{n+1} = 0$ give a solution to $(\mathcal{D}_{\mathcal{Z}})$. We have

$$\max_{y \in \mathbb{R}^n,\ y_{n+1} \in \mathbb{R}} \left\{ y_{n+1} \mid y \leq P^T y - y_{n+1} e,\ y \geq 0,\ y_{n+1} \geq 0 \right\}, \qquad (\mathcal{D}_{\mathcal{Z}})$$

and notice that $e = \overline{y} \leq P^T \overline{y} - \overline{y}_{n+1} e = P^T e - 0e = P^T e = e$ since P is a stochastic matrix and hence the sum of the entries of each column of P is 1 (and therefore the sum of the entries of each row of P^T is 1). Next we need to show the maximum of the admissible y_{n+1}'s does not exceed $\overline{y}_{n+1} = 0$.

Theorem 1.2.C (continued 1)

Proof (continued). Since $y \le P^T y - y_{n+1} e$ then the maximum component of $y \in \mathbb{R}^n$ is less than or equal to the maximum component of $P^T y - y_{n+1} e \in \mathbb{R}^n$. Next, the maximum component of $P^T y$ is less than or equal to the maximum component of $P^T e$ (which is just the maximum row sum of P^T, which is 1 as described above) times the maximum component of y. We therefore have (using subscripts to indicate components of vectors):

$$\max_{1 \le i \le n} \{y_i\} \le \max_{1 \le i \le n} (P^T y - y_{n+1} e) = \max_{1 \le i \le n} (P^T y)_i - y_{n+1}$$

$$\le \max_{1 \le i \le n} (P^T e)_i \max_{1 \le i \le n} y_i - y_{n+1} = \max_{1 \le i \le n} y_i - y_{n+1}$$

where the last equality holds because $\max_{1 \le i \le n} (P^T e)_i = 1$ as described above. Hence $y_{n+1} \le \max_{1 \le i \le n} y_i - \max_{1 \le i \le n} y_i = 0 = \overline{y}_{n+1}$. Therefore $\overline{y} = e$ and $\overline{y}_{n+1} = 0$ is a solution to $(\mathcal{D}_{\mathcal{Z}})$.

Theorem 1.2.C (continued 1)

Proof (continued). Since $y \leq P^T y - y_{n+1} e$ then the maximum component of $y \in \mathbb{R}^n$ is less than or equal to the maximum component of $P^T y - y_{n+1} e \in \mathbb{R}^n$. Next, the maximum component of $P^T y$ is less than or equal to the maximum component of $P^T e$ (which is just the maximum row sum of P^T, which is 1 as described above) times the maximum component of y. We therefore have (using subscripts to indicate components of vectors):

$$\max_{1 \leq i \leq n} \{y_i\} \leq \max_{1 \leq i \leq n} (P^T y - y_{n+1} e) = \max_{1 \leq i \leq n} (P^T y)_i - y_{n+1}$$

$$\leq \max_{1 \leq i \leq n} (P^T e)_i \max_{1 \leq i \leq n} y_i - y_{n+1} = \max_{1 \leq i \leq n} y_i - y_{n+1}$$

where the last equality holds because $\max_{1 \leq i \leq n} (P^T e)_i = 1$ as described above. Hence $y_{n+1} \leq \max_{1 \leq i \leq n} y_i - \max_{1 \leq i \leq n} y_i = 0 = \overline{y}_{n+1}$. Therefore $\overline{y} = e$ and $\overline{y}_{n+1} = 0$ is a solution to $(\mathcal{D}_{\mathcal{Z}})$.

Theorem 1.2.C (continued 2)

Theorem 1.2.C. For stochastic matrix P, the system

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

is feasible. That is, there exists $x \in \mathbb{R}^n$ satisfying the conditions of $(\mathcal{X})$.

Proof (continued). So the dual problem $(\mathcal{D}_{\mathcal{Z}})$ is solvable. Hence, by the Strong Duality Theorem (Theorem 1.2.B), the primal problem $(\mathcal{P}_{\mathcal{Z}})$ is solvable. So by Note 1.2.C, this implies the feasibility of $(\mathcal{Z})$. Then by the Lemma 1.2.B, we have the feasibility of $(\mathcal{X})$, as claimed. □