Section 1.2. Results

Note. As seen in the previous section , the Google Problem involves finding vector $x \in \mathbb{R}^n$ such that:

$$x = Px, \ x \geq 0, \ x \neq 0. \tag{$\mathcal{R}$}$$

Here, $n \times n$ matrix P with nonnegative entries is a "transition matrix" in which the sum of the entries in each column is 1. Such a matrix is also called a *stochastic matrix* (or a "Markov matrix"). If this section we explore the existence of a solution to ($\mathcal{R}$).

Lemma 1.2.A. If P is an $n \times n$ stochastic matrix, then $\det(P - I) = 0$ where I is the $n \times n$ identity matrix.

Note. Lemma 1.2.A guarantees that matrix P has $\lambda = 1$ as an eigenvalue, and hence there is a nonzero vector x (recall that eigenvectors are by definition nonzero) such that $x = Px$. However, we have not yet established that there is an eigenvector x of P satisfying $x > 0$. So we do not yet have a solution to the Google Problem ($\mathcal{R}$). We will prove below that such an eigenvector does in fact exist. Next, we find such a vector for the matrix P associated with network N1 of the previous section.

Note. Consider again matrix P associated the network N1 given in Section 1.1:

$$P = \begin{pmatrix} 0 & 0 & 1 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 \end{pmatrix}.$$

Since P is a stochastic matrix, then we know by Lemma 1.2.A that $\lambda = 1$ is an eigenvalue of P. That is, $\det(P - I) = 0$. Therefore, matrix P is singular (i.e., not invertible) by Theorem 4.3. Determinant Criterion for Invertibility of my online Linear Algebra (MATH 2010) notes on Section 4.2. The Determinant of a Square Matrix. So by "Corollary 2. The Homogeneous Case" of Section 1.6. Homogeneous Systems, Subspaces and Bases from Linear Algebra, there is a nontrivial solution to the system of equations $(P - I)x = 0$ and hence there is an (nonzero) eigenvector x of P associated with $\lambda = 1$. So we consider the augmented matrix $(P - I \mid 0)$ and solve it (using Wolfram Alpha $W\alpha$, for example):

$$(P - I \mid 0) = \left(\begin{array}{ccccc|c} -1 & 0 & 1 & 1/3 & 1/3 & 0 \\ 1 & -1 & 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & -1 & 1/3 & 1/3 & 0 \\ 0 & 1/2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & -1 & 0 \end{array}\right) \underset{W\alpha}{\sim} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -17/3 & 0 \\ 0 & 1 & 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 & -13/3 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So with t as a "free variable," we have that every eigenvector associated with $\lambda = 1$ is of the form $x = t(17/3, 6, 13/3, 3, 1)^T$ where $t \in \mathbb{R}$ and $t \neq 0$. We seek an eigenvector whose components sum to 1 (a type of "normalization" as Shikhman and Müller call it, but not the usual normalization since we do not get a unit vector...at not a unit vector under the usual Euclidean norm on $\mathbb{R}^n$, but there are other norms such as the "ℓ^1 norm" on $\mathbb{R}^n$ under the norm of a vector is the sum of the absolute value of the components). Notice that $17/3 + 6 + 13/3 + 3 + 1 = 20$, then we set $t = 1/20$ and consider the eigenvector $x = (17/60, 6/20, 13/60, 3/20, 1/20)^T = (17/60, 18/60, 13/60, 9/60, 3/60)^T$. Since each component of x is nonnegative then this gives a solution to $(\mathcal{R})$ for matrix P, and hence gives a solution to the Google Problem for the network N1. Comparing the components of x we have the ranking:

Web page i:	1	2	3	4	5
Component x_i:	17/60	18/60	13/60	9/60	3/60
Rank:	II	I	III	IV	V

Notice that web page 2 has the top ranking, even though it only has two incoming hyperlinks. Recall that in Section 1.1 we started with web page 2 ranked as III, behind web pages 1 and 3 (based simply on the number of incoming hyperlinks).

Note 1.2.A. We consider the column vector $e \in \mathbb{R}^n$ with all components of 1, $e = (1, 1, \ldots, 1)^T$. We then have that e^T is the row vector $e^T = (1, 1, \ldots, 1)$. Notice that the components of column vector x sum to 1 if $e^T x = 1$. In these notes, we treat all products as matrix products so we do not use dot product notation (whereas Shikhman and Müller represent all matrix and dot products with $\cdot$). We can then modify the Google Problem for stochastic matrix P to the following:

$$x = Px,\ x \geq 0,\ e^T x = 1. \qquad (\mathcal{X})$$

To establish the "feasibility" (i.e., the existence of the desired vector x) of $(\mathcal{X})$, we consider the "relaxed version" of $(\mathcal{X})$:

$$z \geq Pz,\ z \geq 0,\ e^T z \geq 1. \tag{$\mathcal{Z}$}$$

We claim that a solution z of $(\mathcal{Z})$ implies a solution x of (X) in the following (in which we adopt the "feasibility" terminology).

Lemma 1.2.B. For P is an $n \times n$ stochastic matrix, if $(\mathcal{Z})$ is feasible:

$$z \geq Pz,\ z \geq 0,\ e^T z \geq 1, \tag{$\mathcal{Z}$}$$

then $(\mathcal{X})$ is feasible:

$$x = Px,\ x \geq 0,\ e^T x = 1. \tag{$\mathcal{X}$}$$

Note 1.2.B. We now turn our attention to the feasibility of $(\mathcal{Z})$. We do so by considering the following two questions. We consider $c \in \mathbb{R}^k$, $A \in \mathbb{R}^{m\times k}$, and $b \in \mathbb{R}^m$, and look for $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^m$ satisfying:

$$\min_{u\in\mathbb{R}^k}\{c^T u \mid Au \geq b,\ u \geq 0\}, \tag{$\mathcal{P}$}$$

$$\max_{v\in\mathbb{R}^m}\{b^T v \mid A^T v \leq c,\ v \geq 0\}. \tag{$\mathcal{D}$}$$

These are the "primal" and "dual" linear programming problems, respectively. In applications, $c \in \mathbb{R}^k$, $A \in \mathbb{R}^{m\times k}$, and $b \in \mathbb{R}^m$ contain data.

Theorem 1.2.A. The Weak Duality Theorem.

If $u \in \mathbb{R}^k$ is feasible for $(\mathcal{P})$ and $v \in \mathbb{R}^m$ is feasible for $(\mathcal{D})$, then we have

$$\min_{u\in\mathbb{R}^k}\{c^T u \mid Au \geq b,\ u \geq 0\} \geq \max_{v\in\mathbb{R}^m}\{b^T v \mid A^T v \leq c,\ v \geq 0\}.$$

Note. We now state the Strong Duality Theorem. **For now**, we do not offer a proof but instead reference Chapter 5, "Duality," of S. Boyd and L. Vandenberghe's *Convex Optimization*, Cambridge University Press (2004); a copy is available online on Stephen Boyd's website (accessed 4/23/2021).

Theorem 1.2.B. The Strong Duality Theorem.
$(\mathcal{P})$ is solvable if and only if $(\mathcal{D})$ is solvable and, in this case, the optimal values of $(\mathcal{P})$ and $(\mathcal{D})$ coincide (that is, the minimum value given in $(\mathcal{P})$ equals the maximum value given in $(\mathcal{D})$).

Note 1.2.C. Consider the linear programming problem (where matrix P is $n \times n$)

$$\min_{z \in \mathbb{R}^n} \{0^T z \mid z \geq Pz,\ z \geq 0,\ e^T z \geq 1\}. \tag{$\mathcal{P}_\mathcal{Z}$}$$

Now if there is a $z \in \mathbb{R}^n$ satisfying the conditions $z \geq Pz$, $z \geq 0$, and $e^T z \geq 1$ (that is, if $(\mathcal{Z})$ is feasible) then $(\mathcal{P}_\mathcal{Z})$ is solvable (of course the minimum is 0). Conversely, if $(\mathcal{P}_\mathcal{Z})$ is solvable then there is a vector satisfying the conditions $z \geq Pz$, $z \geq 0$, and $e^T z \geq 1$, so that $(\mathcal{Z})$ is feasible. Hence, $(\mathcal{Z})$ is feasible if and only if $(\mathcal{P}_\mathcal{Z})$ is solvable.

Note. Notice that in $(\mathcal{P}_\mathcal{Z})$, $z \geq Pz$ is equivalent to $z - Pz \geq 0$ or $(I - P)z \geq 0$, so we can combine the two conditions $z \geq Pz$ and $e^T z \geq 1$ of $(\mathcal{P}_\mathcal{Z})$ into a single condition using a partitioned matrix as $\begin{pmatrix} I - P \\ e^T \end{pmatrix} z \geq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; for a discussion of partitioned matrices, see my online notes for Theory of Matrices (MATH 5090) on

Section 3.1. Basic Definitions and Notation. Therefore we can write $(\mathcal{P}_{\mathcal{Z}})$ in primal form as

$$\min_{z\in\mathbb{R}^n}\left\{0^T z \;\middle|\; \begin{pmatrix} I-P \\ e^T \end{pmatrix} z \geq \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ z \geq 0 \right\}.$$

Notice that in the notation of $(\mathcal{P})$, we have here that $A = \begin{pmatrix} I-P \\ e^T \end{pmatrix}$ is an $(n+1)\times n$ matrix. So $A^T = \begin{pmatrix} I-P \\ e^T \end{pmatrix}^T$ is $n \times (n+1)$ and the dual problem then involves vectors in $\mathbb{R}^{n+1}$. We consider such a vector as partitioned into a vector $y \in \mathbb{R}^n$ and a vector $y_{n+1} \in \mathbb{R}^1$. Also in the notation of $(\mathcal{P})$, we have here $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$ (here "0" denotes a vector in $\mathbb{R}^n$), so $b^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$ is an $1 \times (n+1)$ matrix. So the dual problem corresponding the problem here is

$$\max_{y\in\mathbb{R}^n,\ y_{n+1}\in\mathbb{R}}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} y \\ y_{n+1} \end{pmatrix} \;\middle|\; \begin{pmatrix} I-P \\ e^T \end{pmatrix}^T \begin{pmatrix} y \\ y_{n+1} \end{pmatrix} \leq 0,\ y \geq 0,\ y_{n+1} \geq 0 \right\}.$$

Notice that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} y \\ y_{n+1} \end{pmatrix} = y_{n+1}$ and

$$\begin{pmatrix} I-P \\ e^T \end{pmatrix}^T \begin{pmatrix} y \\ y_{n+1} \end{pmatrix} = \left((I-P)^T\ \ e\right) \begin{pmatrix} y \\ y_{n+1} \end{pmatrix}$$

$$= (I-P)^T y + e y_{n+1} = Iy - P^T y + y_{n+1} e = y - P^T y + y_{n+1} e,$$

so that the dual problem can be simplified to

$$\max_{y\in\mathbb{R}^n,\ y_{n+1}\in\mathbb{R}} \left\{ y_{n+1} \mid y \leq P^T y - y_{n+1} e,\ y \geq 0,\ y_{n+1} \geq 0 \right\}. \qquad (\mathcal{D}_{\mathcal{Z}})$$

Theorem 1.2.C. For stochastic matrix P, the system

$$x = Px,\ x \geq 0,\ e^T x = 1. \tag{$\mathcal{X}$}$$

is feasible. That is, there exists $x \in \mathbb{R}^n$ satisfying the conditions of $(\mathcal{X})$.

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