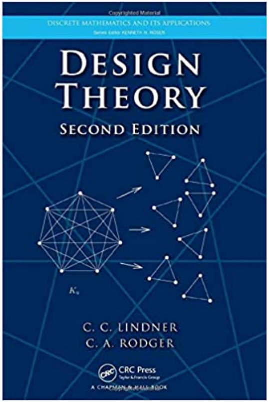
Design Theory

Chapter 1. Steiner Triple Systems

1.1. The Existence Problem—Proofs of Theorems



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Lemma 1.1.A

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Proof. If (S, T) is a Steiner triple system of order v, then any triple, say $\{a, b, c\}$, contains three 2-element subsets, namely $\{a, b\}$, {a,c}, $\{b, c\}$, and S contains $\binom{v}{2} = \frac{v(v-1)}{2}$ 2-element subsets. Since every pair of distinct elements of S occurs together in exactly one triple of T, then $3|T| = \binom{v}{2}$, or

$$|T| = v(v-1)/6. \qquad (*)$$

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For any $x \in S$, set $T(x) = \{t \setminus \{x\} \mid x \in t \in T\}$ (so $T(x)$ consists of all 2-element subsets of S that do not include x). Then $T(x)$ partitions $S \setminus \{x\}$ into 2-element subsets. Since $|S \setminus \{x\}| = v - 1$, then we must have that $v - 1$ is even (since we have partitioned this set of size $v - 1$ into subsets of size 2).

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Lemma 1.1.A (continued)

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Proof (continued). That is, v must be odd and so $v \equiv 1, 3$, or 5 (mod 6). But for $v \equiv 5 \pmod 6$, we have by $(*)$ that $|T| = v(v-1)/6$ is not an integer (since $v(v-1) \equiv 2 \pmod 6$), so we cannot have $v \equiv 5$ (mod 6). So necessary conditions for a $STS(v)$ to exist is that $v \equiv 1$ or 3 (mod 6), as claimed. □