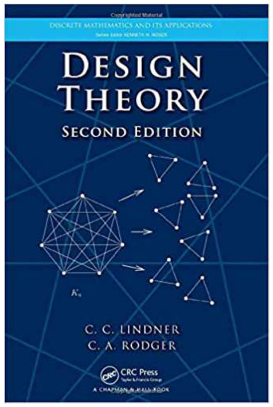
# Design Theory

**Chapter 1. Steiner Triple Systems**

1.2. $v \equiv 3 \pmod 6$: The Bose Construction—Proofs of Theorems








## Theorem 1.2.A

**Theorem 1.2.A.** A Steiner triple system of all orders $v \equiv 3 \pmod 6$ exist.

**Proof.** We use Exercise 1.1.4 which states: Let $S$ be a set of size $T$ by a set of 3-element subsets of $S$. Furthermore, suppose that

(a) each pair of distinct elements of $S$ belongs to *at least* one triple in $T$, and

(b) $|T| \le v(v-1)/6$.

Then $(S, T)$ is a Steiner triple system. The number of triples in $T$ of Type 1 is $2n+1$. For the type 2 triples, there are $\binom{2n+1}{2} = \frac{(2n+1)(2n)}{2} = n(2n+1)$ choice for $i$ and $j$ by the Fundamental Counting Principle (see my online notes for Foundations of Probability and Statistics-Calculus (MATH 2050) on Section 2.2. Counting Methods, Note 2.2.A, and for Applied Combinatorics and Problem Solving (MATH 3340) on Section 1.1. The Fundamental Counting Principle). Each choice of $i$ and $j$ yields three triples of Type 2 (see Figure 1.4), so that there are $3n(2n+1)$ triples of Type 2.





## Theorem 1.2.A (continued 1)

**Theorem 1.2.A.** A Steiner triple system of all orders $v \equiv 3 \pmod 6$ exist.

**Proof (continued).** So the total number of triples is

$$|T| = (2n+1) + 3n(2n+1) = (2n+1)(3n+1)$$

$$= (6n+3)(6n+2)/6 = v(v-1)/6.$$

So (b) of Exercise 1.4 is satisfied.

Let $(a, b)$ and $(c, d)$ be a pair of elements of $S$. We consider three cases. Case 1. Suppose that $a = c$. The $\{(a,1),(a,2),(a,3)\}$ is a Type 1 triple in $T$ and contains $(a, b)$ and $(c, d)$.
Case 2. Suppose that $b = d$. Then $a \ne c$ (otherwise the elements of $S$ are not distinct) and so the Type 2 triple $\{(a,b),(c,b),(a \circ c, z)\} \in T$ where $z = 1$ if $b = d = 3$, $z = 2$ is $b = d = 1$, and $z = 3$ if $b = d = 2$, and this triple contains $(a, b)$ and $(c, d)$.





## Theorem 1.2.A (continued 2)

**Theorem 1.2.A.** A Steiner triple system of all orders $v \equiv 3 \pmod 6$ exist.

**Proof (continued).** Case 3. Suppose that $a \ne c$ and $b \ne d$. If $b = 1$ and $d = 2$ then in quasigroup $(Q, \circ)$ we have $a \circ i = c$ for some $i \in Q$. Since $(S\circ)$ is idempotent and $a \ne c$ (and solutions to such equations are unique), we must have that $i \ne a$. Therefore $\{(a,1),(i,1),(a \circ i, 2)\}$ is a Type 2 triple in $T$ and contains $(a, b)$ and $(c, d)$. The other possible values of $b$ and $d$ are similar.

In each case, the pair of elements $(a, b)$ and $(c, d)$ of elements of $S$ belong to at least one element of $T$. So, by Exercise 1.1.4, $(S, T)$ is a Steiner triple system. □