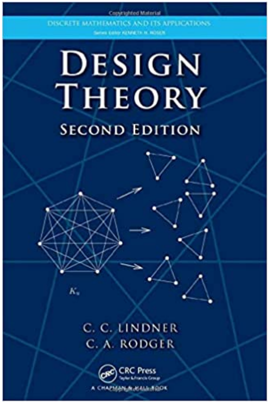
Design Theory

Chapter 1. Steiner Triple Systems

1.2. $v \equiv 3 \pmod 6$: The Bose Construction—Proofs of Theorems



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Theorem 1.2.A

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Proof. We use Exercise 1.1.4 which states: Let S be a set of size T by a set of 3-element subsets of S. Furthermore, suppose that

(a) each pair of distinct elements of S belongs to *at least* one triple in T, and

(b) $|T| \leq v(v-1)/6$.

Then (S, T) is a Steiner triple system.

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Then (S, T) is a Steiner triple system. The number of triples in T of Type 1 is $2n+1$. For the type 2 triples, there are $\binom{2n+1}{2} = \frac{(2n+1)(2n)}{2} = n(2n+1)$ choice for i and j by the Fundamental Counting Principle (see my online notes for Foundations of Probability and Statistics-Calculus (MATH 2050) on Section 2.2. Counting Methods, Note 2.2.A, and for Applied Combinatorics and Problem Solving (MATH 3340) on Section 1.1. The Fundamental Counting Principle). Each choice of i and j yields three triples of Type 2 (see Figure 1.4), so that there are $3n(2n+1)$ triples of Type 2.

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Proof (continued). So the total number of triples is

$$|T| = (2n+1) + 3n(2n+1) = (2n+1)(3n+1)$$

$$= (6n+3)(6n+2)/6 = v(v-1)/6.$$

So (b) of Exercise 1.4 is satisfied.

Let (a, b) and (c, d) be a pair of elements of S. We consider three cases. Case 1. Suppose that $a = c$. The $\{(a, 1), (a, 2), (a, 3)\}$ is a Type 1 triple in T and contains (a, b) and (c, d).

Case 2. Suppose that $b = d$. Then $a \neq c$ (otherwise the elements of S are not distinct) and so the Type 2 triple $\{(a, b), (c, b), (a \circ c, z)\} \in T$ where $z = 1$ if $b = d = 3$, $z = 2$ is $b = d = 1$, and $z = 3$ if $b = d = 2$, and this triple contains (a, b) and (c, d).

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Theorem 1.2.A (continued 2)

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Proof (continued). Case 3. Suppose that $a \neq c$ and $b \neq d$. If $b = 1$ and $d = 2$ then in quasigroup $(Q, \circ)$ we have $a \circ i = c$ for some $i \in Q$. Since $(S \circ)$ is idempotent and $a \neq c$ (and solutions to such equations are unique), we must have that $i \neq a$. Therefore $\{(a, 1), (i, 1), (a \circ i, 2)\}$ is a Type 2 triple in T and contains (a, b) and (c, d). The other possible values of b and d are similar.

In each case, the pair of elements (a, b) and (c, d) of elements of S belong to at least one element of T. So, by Exercise 1.1.4, (S, T) is a Steiner triple system. □

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