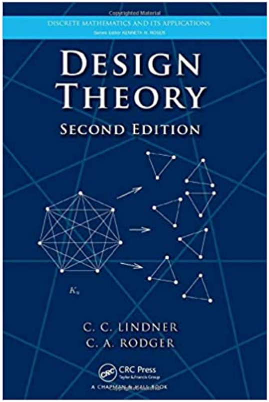
Design Theory

Chapter 1. Steiner Triple Systems

1.5. Quasigroups with Holes and Steiner Triple Systems—Proofs of Theorems



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Proof. Let $S = \{1,2,\ldots,2n+1\}$. If $2n+1 \equiv 1$ or $3 \pmod 6$, then let (S,B) be a Steiner triple system of order $2n+1$ (which exists by the Bose Construction of Section 1.2 for $v \equiv 3 \pmod 6$ and which exists by the Skolem Construction of Section 1.3 for $v \equiv 1 \pmod 6$). If $2n+1 \equiv 5 \pmod 6$ then let (S,B) be a PBD$(2n+1)$ with exactly on block, say b, of size 5 and the rest of size 3 (which exists by the construction of Section 1.4 and Exercises 1.4.6 and 1.4.7). By Exercise 1.5.A, the symbols in these structures can be renamed (if necessary) so that the only triples containing the symbol $v = 2n+1$ are: $\{1,2,2n+1\}$, $\{2,3,2n+1\}$,$\ldots$, $\{2n-1,2n,2n+1\}$.

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Theorem 1.5.5 (continued 1)

Proof (continued). We now define a quasigroup $(Q, \circ) = (\{1, 2, \ldots, 2n\}, \circ)$ as follows:

(a) For each $h \in H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$ let $(h, \circ)$ be a subquasigroup of $(Q, \circ)$;

(b) for $1 \leq i \neq j \leq 2n$, $\{i, j\} \notin H$ and $\{i, j\} \not\subseteq b$, let $\{i, j, k\}$ be the triple in B containing symbols i and j and define $i \circ j = k = j \circ i$; and

(c) if $2n + 1 \equiv 5 \pmod 6$ then let $(b, \otimes)$ be an idempotent commutative quasigroup of order 5 (as given in Example 1.2.2) and for each $\{i, j\} \subseteq b$ define $i \circ j = i \otimes j = j \circ i$.

By (a), the holes of the quasigroup $(Q, \circ)$ are filled symmetrically, guaranteeing commutivity.

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Theorem 1.5.5. For all $n \geq 3$, there exists a commutative quasigroup of order $2n$ with holes $H = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$.

Proof (continued). By (b), the binary operation $\circ$ is defined on every pair of symbols contained in some triple but *not* in a hole, and this is done in such a way that commutivity holds (this completes the construction in the event that a Steiner triple system is used to construct $(Q,\circ)$). In (c), $\circ$ is defined on the remaining pairs of symbols (those five in block b). Since a commutative quasigroup of order 5 is used, then we have that $\circ$ is commutative on these elements as well. Therefore, $\circ$ is defined on all elements of Q and the uniqueness of triples containing pairs of elements in the Steiner triple system or in the pairwise balanced design insures (along with the use of the quasigroup of order 5) the unique appearance of every symbol in each row and each column (that is, $(Q,\circ)$ is a quasigroup). □

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