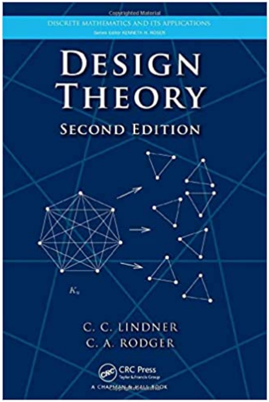
Design Theory

Chapter 1. Steiner Triple Systems

1.6. The Wilson Construction—Proofs of Theorems



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Lemma 1.6.A

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Proof. If cycles C_1 and C_2 are of even length, we can take alternate edges (i.e., every-other-edge) of C_1 and C_2 as two of the 1-factors, and F as the third 1-factor.

If cycles C_1 and C_2 are of odd length k, then we first partition C_1 into three partial 1-factors P_1, P_2, and P_3 (for example, we could take $\lfloor k/2 \rfloor$ alternate edges of the cycle for P_1 and then for P_2, and finally take the single remaining edge for P_3). Next, partition C_2 into three partial 1-factors, $P_1\alpha$, $P_2\alpha$, and $P_3\alpha$ defined by $\{x\alpha, y\alpha\} \in P_i\alpha$ if and only if $\{x, y\} \in P_i$ (that is, we use the graph isomorphism α in the definition of biwheel to project the 1-factors of C_1 onto C_2).

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Lemma 1.6.A (continued)

Proof (continued). There are two edges in C_1 incident to each vertex of C_1. One of these edges in in 1-factor P_i and the other is in 1-factor P_j where $i \neq j$. So the third edge incident to a vertex of C_1 (such an edge is in F) could be assigned to 1-factor P_k where $k \notin \{i, j\}$. Since 1-factors $P_1\alpha$, $P_2\alpha$, $P_3\alpha$ of C_2 are projections 1-factors P_1, P_2, P_3 or C_1 we just need to make sure that, for each third edge incident a vertex of C_1, the other end (in C_2) does not conflict with the 1-factors $P_1\alpha$, $P_2\alpha$, $P_3\alpha$ containing the edges incident to this other end in C_2. This is accomplished assigning the edges of F to 1-factors as follows (and then taking unions of the resulting 1-factors which are "independent" to produce a total of only three 1-factors).

Partition F into 1-factors F_1, F_2, and F_3 defined by $\{x, x\alpha\} \in F_i$ if and only if $\{a, x\}$ and $\{x, b\}$ in C_1 do *not* belong to P_i. Then for $i \in \{1, 2, 3\}$ we have that $P_i \cup P_i\alpha \cup F_i$ is a 1-factor of the biwheel. These three 1-factors give a 1-factorization of the biwheel $C_1 \cup C_2 \cup F$, as claimed. □

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