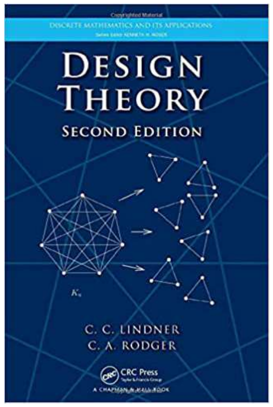
# Design Theory

**Chapter 2. $\lambda$-Fold Triple Systems**

2.2. The Existence of Idempotent Latin Squares—Proofs of Theorems








## Lemma 2.2.A

**Lemma 2.2.A.** Let $(\{1,2,\ldots,n-1\},\circ)$ be a quasigroup of order $n-1$. The order-$n$ binary algebraic structure $\langle\{1,2,\ldots,n-1,n\},*\rangle$ that results from stripping a transversal of the quasigroup of order $n-1$ is itself a quasigroup.

**Proof.** We need to show that each row of the Cayley table for the order-$n$ binary algebraic structure contains each of the symbols $\{1,2,\ldots,n\}$ exactly once, and similarly for each column. Consider row $i$ of the quasigroup of order $n-1$, where $1 \le i \le n-1$. This row contains each of the elements in $\{1,2,\ldots,n-1\}$ exactly once. For $T$ any transversal, $T$ contains one cell of row $i$, say cell $(i,j)$. In stripping transversal $T$, cell $(i,j)$ gets the "new" symbol $n$ (Step 1) and the new cell $(i,n)$ gets the symbol from cell $(i,j)$ in the original quasigroup (Step 1 if $1 \le i \le n-1$). So row $i$ of the Cayley table for the order-$n$ binary algebraic structure contains each symbol of $\{1,2,\ldots,n\}$ exactly once.





## Lemma 2.2.A (continued)

**Proof.** Now for row $n$, the entries in the first $n-1$ cells are the entries in the transversal and these are (by the definition of transversal) the symbols $1,2,\ldots,n-1$. By Step 3 the entry in cell $(n,n)$ is $n$ so that row $n$ contains each symbol of $\{1,2,\ldots,n\}$ exactly once.

Consider column $j$ of the quasigroup of order $n-1$. This column contains each of the elements in $\{1,2,\ldots,n-1\}$ exactly once. For $T$ any transversal, $T$ contains one cell of column $j$, say cell $(i,j)$. In stripping transversal $T$, cell $(i,j)$ gets the "new" symbol $n$ (Step 1) and the new cell $(n,j)$ gets the symbol from cell $(i,j)$ in the original quasigroup (Step 1 if $1 \le j \le n-1$). So column $j$ of the Cayley table for the order-$n$ binary algebraic structure contains each symbol of $\{1,2,\ldots,n\}$ exactly once. Now for column $n$, the entries in the first $n-1$ cells are the entries in the transversal and these are (by the definition of transversal) the symbols $1,2,\ldots,n-1$. By Step 3 the entry in cell $(n,n)$ is $n$ so that column $n$ contains each symbol of $\{1,2,\ldots,n\}$ exactly once. Hence, the order-$n$ binary algebraic structure is actually a quasigroup, as claimed. □





## Theorem 2.2.3

**Theorem 2.2.3.** For all $n \neq 2$, there exists an idempotent quasigroup of order $n$.

**Proof.** If $n$ is odd, then an idempotent (commutative) latin square exists of order $n$ by Exercise 1.2.3(a,iii); this latin square is based on rearranging the Cayley table for $\mathbb{Z}_n$ and renaming the marginal entries. We don't really distinguish between latin squares and quasigroups (see page 4 or the text book: "As far as we are concerned a quasigroup is just a latin square with a headline and a sideline."). We consider the transversal $T = \{(1,2),(2,3),\ldots,(n-1,n),(n,1)\}$ in an odd order $n \ge 3$ idempotent quasigroup. Since the quasigroup is based on $\mathbb{Z}_n$, then the entries in this transversal are all different.

# Theorem 2.2.3 (continued)

**Theorem 2.2.3.** For all $n \neq 2$, there exists an idempotent quasigroup of order $n$.

**Proof (continued).** We create an even order $n+1$ quasigroup by stripping the transversal $T$ of the order $n$ quasigroup. Since $T$ includes no cells of the form $(i, i)$ then the entries in these cells remain the same for $1 \leq i \leq n-1$ (by Step 2) and in cell $(n+1, n+1)$ the entry is $n+1$ (by Step 3). So the $n+1$ order quasigroup is idempotent, as needed. Therefore, an even order idempotent quasigroup exists for all even $n > 2$. □