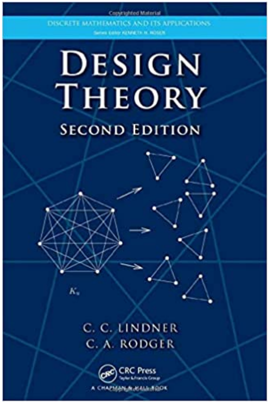
Design Theory

Chapter 2. λ-Fold Triple Systems

2.2. The Existence of Idempotent Latin Squares—Proofs of Theorems



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Proof. We need to show that each row of the Cayley table for the order-n binary algebraic structure contains each of the symbols $\{1, 2, \ldots, n\}$ exactly once, and similarly for each column. Consider row i of the quasigroup of order $n-1$, where $1 \leq i \leq n-1$. This row contains each of the elements in $\{1, 2, \ldots, n-1\}$ exactly once. For T any transversal, T contains one cell of row i, say cell (i, j).

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Lemma 2.2.A (continued)

Proof. Now for row n, the entries in the first $n-1$ cells are the entries in the transversal and these are (by the definition of transversal) the symbols $1, 2, \ldots, n-1$. By Step 3 the entry in cell (n, n) is n so that row n contains each symbol of $\{1, 2, \ldots, n\}$ exactly once.

Consider column j of the quasigroup of order $n-1$. This column contains each of the elements in $\{1, 2, \ldots, n-1\}$ exactly once. For T any transversal, T contains one cell of column j, say cell (i, j).

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Consider column j of the quasigroup of order $n-1$. This column contains each of the elements in $\{1, 2, \ldots, n-1\}$ exactly once. For T any transversal, T contains one cell of column j, say cell (i, j). In stripping transversal T, cell (i, j) gets the "new" symbol n (Step 1) and the new cell (n, j) gets the symbol from cell (i, j) in the original quasigroup (Step 1 if $1 \leq j \leq n-1$). So column j of the Cayley table for the order-n binary algebraic structure contains each symbol of $\{1, 2, \ldots, n\}$ exactly once. Now for column n, the entries in the first $n-1$ cells are the entries in the transversal and these are (by the definition of transversal) the symbols $1, 2, \ldots, n-1$. By Step 3 the entry in cell (n, n) is n so that column n contains each symbol of $\{1, 2, \ldots, n\}$ exactly once. Hence, the order-n binary algebraic structure is actually a quasigroup, as claimed. □

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Theorem 2.2.3

Theorem 2.2.3. For all $n \neq 2$, there exists an idempotent quasigroup of order n.

Proof. If n is odd, then an idempotent (commutative) latin square exists of order n by Exercise 1.2.3(a,iii); this latin square is based on rearranging the Cayley table for $\mathbb{Z}_n$ and renaming the marginal entries. We don't really distinguish between latin squares and quasigroups (see page 4 or the text book: "As far as we are concerned a quasigroup is just a latin square with a headline and a sideline."). We consider the transversal $T = \{(1,2),(2,3),\ldots,(n-1,n),(n,1)\}$ in an odd order $n \geq 3$ idempotent quasigroup. Since the quasigroup is based on $\mathbb{Z}_n$, then the entries in this transversal are all different.

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Theorem 2.2.3 (continued)

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Proof (continued). We create an even order $n+1$ quasigroup by stripping the transversal T of the order n quasigroup. Since T includes no cells of the form (i, i) then the entries in these cells remain the same for $1 \leq i \leq n-1$ (by Step 2) and in cell $(n+1, n+1)$ the entry is $n+1$ (by Step 3). So the $n+1$ order quasigroup is idempotent, as needed. Therefore, an even order idempotent quasigroup exists for all even $n > 2$. □