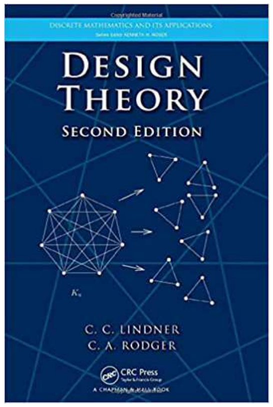
Design Theory

Chapter 2. λ-Fold Triple Systems

2.3. 2-Fold Triple Systems—Proofs of Theorems



Theorem 2.3.7

Theorem 2.3.7. The spectrum for 2-fold triple systems is precisely the of all $v \equiv 0$ or $1 \pmod 3$.

Proof. We use Exercise 2.3.5 which states that if T contains at least $v(v-1)/3$ triples and if every pair of elements of S belongs to at least two triples of T, then (S, T) is a 2-fold triple system of order v. In Note 2.3.A, we see in the case $v = 3n$ that there are $2n$ Type 1 triples and $6\binom{n}{2} = \frac{6n(n-1)}{2} = 3n(n-1)$ Type 2 triples. The total number of triples in this case is then $2n + 3n(n-1) = 3n^2 - n = n(3n-1)$ $= (3n)(3n-1)/3 = v(v-1)/3$. In Note 2.3.B, we see in the case $v = 3n+1$ that there are $4n$ Type 1 triples and $6\binom{n}{2} = 3n(n-1)$ Type 2 triples. The total number of triples in this case is then $4n + 3n(n-1) = 3n^2 + n = n(3n+1) = (3n)(3n+1)/3 = v(v-1)/3$. By Exercise 2.3.5, if we can show that every possible pair is present twice in the collection of triples, then the claim will follow.

Theorem 2.3.7 (continued 1)

Proof (continued). Consider the case $v = 3n$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form (i_1, j), (i_2, j), where $i_1 \neq i_2$, $i_1, i_2 \in Q$, and $j \in \{1, 2, 3\}$, appears twice in the Type 2 triples (in the first two positions of the triples as given in Note 2.3.A). Every pair of the form $(i_1, 1)$, $(i_2, 2)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the first two Type 2 triples as given in Note 2.3.A); since $(Q, \circ)$ is idempotent then in the Type 2 triples $x \circ y, y \circ x \notin \{x, y\}$ so that there is not another repetition of the pairs (i, j_1), (i, j_2) where $j_1 = 1$ and $j_2 = 2$ (this is where the idempotent property is needed). Similarly, every pair of the form $(i_1, 2)$, $(i_2, 3)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the second two Type 2 triples as given in Note 2.3.A). Also, every pair of the form $(i_1, 3)$, $(i_2, 1)$, where $i_1, i_2 \in Q$ and $i_1 \neq i_2$, appears exactly twice in the Type 2 triples (in the third two Type 2 triples as given in Note 2.3.A). This includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 0 \pmod 3$ exists.

Theorem 2.3.7 (continued 2)

Theorem 2.3.7. The spectrum for 2-fold triple systems is precisely the of all $v \equiv 0$ or $1 \pmod 3$.

Proof (continued). Consider the case $v = 3n + 1$. Every pair of the form (i, j_1), (i, j_2), where $i \in Q$ and $j_1, j_2 \in \{1, 2, 3\}$ with $j_1 \neq j_2$, appears twice in the Type 1 triples. Every pair of the form ∞, (i, j), where $i \in Q$ and $j \in \{1, 2, 3\}$, appears twice in the Type 1 triples. As in the case $v = 3n$, the Type 2 triples cover the remaining types of pairs exactly twice (namely, (i_1, j_1), (i_2, j_2) where $i_1, i_2 \in Q$, $i_1 \neq i_2$, $j_1, j_2 \in \{1, 2, 3\}$). Since this includes all possible types of pairs, so that a 2-fold triple system of order $v \equiv 1 \pmod 3$ exists.

Therefore, a 2-fold triple system of order v exists if and only if $v \equiv 0$ or $1 \pmod 3$. □